The Integral Test Part III: Estimations and Errors

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Integral Test

Theorem: [The Integral Test]

Assume that

f is continuous on $[1, \infty)$,
$f(x) > 0$ on $[1, \infty)$,
f is decreasing on $[1, \infty)$, and
$a_k = f(k)$.

Then

1) If $S_n = \sum_{k=1}^{n} a_k$, then for all $n \in \mathbb{N}$,

$$\int_1^{n+1} f(x)\,dx \leq S_n \leq a_1 + \int_1^n f(x)\,dx.$$

2) $\sum_{k=1}^{\infty} a_k$ converges if and only if $\int_1^\infty f(x)\,dx$ converges.

3) If $\sum_{k=1}^{\infty} a_k$ converges, with $S = \sum_{k=1}^{\infty} a_k$, then

$$\int_1^\infty f(x)\,dx \leq \sum_{k=1}^{\infty} a_k \leq a_1 + \int_1^\infty f(x)\,dx$$

and

$$\int_{n+1}^\infty f(x)\,dx \leq S - S_n \leq \int_n^\infty f(x)\,dx.$$

Harmonic Series

Example: Show that

$$\ln(n+1) \leq \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} \leq 1 + \ln(n)$$

for each $n \in \mathbb{N}$.

Let $f(x) = \frac{1}{x}$, $a_k = f(k)$ and $S_n = \sum_{k=1}^{n} \frac{1}{k} = \frac{1}{1} + \frac{1}{2} + \cdots + \frac{1}{n}$, then by the Integral Test we have

$$\int_1^{n+1} \frac{1}{x}\,dx \leq S_n \leq a_1 + \int_1^{n} \frac{1}{x}\,dx$$

We also know that

$$\int_1^{n+1} \frac{1}{x}\,dx = \ln(n+1) - \ln(1) = \ln(n+1)$$

and

$$\int_1^{n} \frac{1}{x}\,dx = \ln(n) - \ln(1) = \ln(n)$$

Since $a_1 = \frac{1}{1} = 1$, we get

$$\ln(n+1) \leq \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} \leq 1 + \ln(n)$$

for each $n \in \mathbb{N}$.

Harmonic Series

Problem: We know that $\sum_{k=1}^{\infty} \frac{1}{k}$ diverges to ∞. How large must n be so that

$$\sum_{k=1}^{n} \frac{1}{k} = \frac{1}{1} + \frac{1}{2} + \cdots + \frac{1}{n} > 100?$$

We know that

$$\ln(n+1) \leq \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} \leq 1 + \ln(n)$$

so we can choose n large enough so that

$$\ln(n+1) > 100$$

This would mean that

$$n + 1 > e^{100} \Rightarrow n > e^{100} - 1$$

Question: Could a modern computer add up enough terms in the series one at a time to reach 100?

$$e^{100} \cong 2.7 \times 10^{43}$$

More Examples

Example: The p-Series Test shows that the series $\sum_{n=1}^{\infty} \frac{1}{n^4}$ converges. Let

$$S_k = \sum_{n=1}^{k} \frac{1}{n^4} \quad \text{and} \quad S = \sum_{n=1}^{\infty} \frac{1}{n^4}.$$

Estimate the error in using the first 100 terms in the series to approximate S. That is, estimate $|S - S_{100}|$.

Since all the terms are positive we have that $|S - S_{100}| = S - S_{100}$. and from the Integral Test we get that

$$\int_{101}^{\infty} \frac{1}{x^4}\, dx \leq S - S_{100} \leq \int_{100}^{\infty} \frac{1}{x^4}\, dx.$$

For any $m \in \mathbb{N}$, we have that

$$\begin{aligned} \int_{m}^{\infty} \frac{1}{x^4}\, dx &= \lim_{b\to\infty} \int_{m}^{b} \frac{1}{x^4}\, dx \\ &= \lim_{b\to\infty} -\frac{1}{3x^3}\Big|_m^b \\ &= \lim_{b\to\infty} \frac{1}{3m^3} - \frac{1}{3b^3} = \frac{1}{3m^3} \end{aligned}$$

More Examples

Example Cont'd: Letting $m = 101$ and $m = 100$ respectively tells us that

$$\frac{1}{3(101)^3} \leq S - S_{100} \leq \frac{1}{3(100)^3}$$

or

$$3.2353 \times 10^{-7} \leq S - S_{100} \leq 3.3333 \times 10^{-7}.$$

Calculating S_{100} gives $S_{100} = 1.082322905$ up to 9 decimal places and hence

$$1.082323229 \leq S \leq 1.082323238$$

In fact, it is actually known that

$$S = \sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90} = 1.082323234$$

which does indeed lie within our range.