The Integral Test Part III: Estimations and Errors

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**Integral Test**

**Theorem: [The Integral Test]**

Assume that

- $f$ is continuous on $[1, \infty)$,
- $f(x) > 0$ on $[1, \infty)$,
- $f$ is decreasing on $[1, \infty)$, and
- $a_k = f(k)$.

Then

1) If $S_n = \sum_{k=1}^{n} a_k$, then for all $n \in \mathbb{N}$,

$$\int_{1}^{n+1} f(x) \, dx \leq S_n \leq a_1 + \int_{1}^{n} f(x) \, dx.$$  

2) $\sum_{k=1}^{\infty} a_k$ converges if and only if $\int_{1}^{\infty} f(x) \, dx$ converges.

3) If $\sum_{k=1}^{\infty} a_k$ converges, with $S = \sum_{k=1}^{\infty} a_k$, then

$$\int_{1}^{\infty} f(x) \, dx \leq \sum_{k=1}^{\infty} a_k \leq a_1 + \int_{1}^{\infty} f(x) \, dx$$

and

$$\int_{n+1}^{\infty} f(x) \, dx \leq S - S_n \leq \int_{n}^{\infty} f(x) \, dx.$$
Harmonic Series

**Example:** Show that

\[
\ln(n + 1) \leq \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} \leq 1 + \ln(n)
\]

for each \( n \in \mathbb{N} \).

Let \( f(x) = \frac{1}{x} \), \( a_k = f(k) \) and \( S_n = \sum_{k=1}^{n} \frac{1}{k} = \frac{1}{1} + \frac{1}{2} + \cdots + \frac{1}{n} \), then by the Integral Test we have

\[
\int_{1}^{n+1} \frac{1}{x} \, dx \leq S_n \leq a_1 + \int_{1}^{n} \frac{1}{x} \, dx
\]

We also know that

\[
\int_{1}^{n+1} \frac{1}{x} \, dx = \ln(n + 1) - \ln(1) = \ln(n + 1)
\]

and

\[
\int_{1}^{n} \frac{1}{x} \, dx = \ln(n) - \ln(1) = \ln(n)
\]

Since \( a_1 = \frac{1}{1} = 1 \), we get

\[
\ln(n + 1) \leq \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} \leq 1 + \ln(n)
\]

for each \( n \in \mathbb{N} \).
Harmonic Series

**Problem:** We know that \[ \sum_{k=1}^{\infty} \frac{1}{k} \] diverges to \( \infty \). How large must \( n \) be so that

\[ \sum_{k=1}^{n} \frac{1}{k} = \frac{1}{1} + \frac{1}{2} + \cdots + \frac{1}{n} > 100? \]

We know that

\[ \ln(n + 1) \leq \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} \leq 1 + \ln(n) \]

so we can choose \( n \) large enough so that

\[ \ln(n + 1) > 100 \]

This would mean that

\[ n + 1 > e^{100} \Rightarrow n > e^{100} - 1 \]

**Question:** Could a modern computer add up enough terms in the series one at a time to reach 100?

\[ e^{100} \approx 2.7 \times 10^{43} \]
Example: The $p$-Series Test shows that the series $\sum_{n=1}^{\infty} \frac{1}{n^4}$ converges. Let

$$S_k = \sum_{n=1}^{k} \frac{1}{n^4} \quad \text{and} \quad S = \sum_{n=1}^{\infty} \frac{1}{n^4}.$$ 

Estimate the error in using the first 100 terms in the series to approximate $S$. That is, estimate $|S - S_{100}|$.

Since all the terms are positive we have that $|S - S_{100}| = S - S_{100}$. and from the Integral Test we get that

$$\int_{101}^{\infty} \frac{1}{x^4} \, dx \leq S - S_{100} \leq \int_{100}^{\infty} \frac{1}{x^4} \, dx.$$ 

For any $m \in \mathbb{N}$, we have that

$$\int_{m}^{\infty} \frac{1}{x^4} \, dx = \lim_{b \to \infty} \int_{m}^{b} \frac{1}{x^4} \, dx$$

$$= \lim_{b \to \infty} -\frac{1}{3x^3} \bigg|_{m}^{b}$$

$$= \lim_{b \to \infty} \frac{1}{3m^3} - \frac{1}{3b^3} = \frac{1}{3m^3}$$
Example Cont’d: Letting $m = 101$ and $m = 100$ respectively tells us that
\[
\frac{1}{3(101)^3} \leq S - S_{100} \leq \frac{1}{3(100)^3}
\]
or
\[
3.2353 \times 10^{-7} \leq S - S_{100} \leq 3.3333 \times 10^{-7}.
\]
Calculating $S_{100}$ gives $S_{100} = 1.082322905$ up to 9 decimal places and hence
\[
1.082323229 \leq S \leq 1.082323238
\]
In fact, it is actually known that
\[
S = \sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90} = 1.082323234
\]
which does indeed lie within our range.