Divergence Test

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Convergence for Series

Definition: [Convergent Series]

Given a sequence $\{a_n\} = \{a_1, a_2, a_3, \ldots\}$, we define the *$k$th partial sum, S_k* of the series

$$\sum_{n=1}^{\infty} a_n$$

by

$$S_k = a_1 + a_2 + \cdots + a_k = \sum_{n=1}^{k} a_n.$$

We say that the series $\sum_{n=1}^{\infty} a_n$ *converges* if the sequence of partial sums $\{S_k\}$ converges. In this case, we write

$$\sum_{n=1}^{\infty} a_n = \lim_{k\to\infty} S_k.$$

Otherwise, we say that the series *diverges* and the sum has no defined value.

Divergence Test

Problem: Does $\sum_{n=1}^{\infty} \frac{n}{n+1}$ converge?

Observation: Assume that $\sum_{n=1}^{\infty} a_n$ converges to L and that

$$S_k = a_1 + a_2 + \cdots + a_k = \sum_{n=1}^{k} a_n.$$

Then

$$S_{k+1} - S_k = \sum_{n=1}^{k+1} a_n - \sum_{n=1}^{k} a_n = a_{k+1}.$$

It follows that

$$\begin{aligned} \lim_{k\to\infty} a_k &= \lim_{k\to\infty} a_{k+1} = \lim_{k\to\infty} S_{k+1} - S_k = \lim_{k\to\infty} S_{k+1} - \lim_{k\to\infty} S_k \\ &= L - L = 0. \end{aligned}$$

If $a_n = \frac{n}{n+1}$, then $a_n \to 1 \neq 0$, so $\sum_{n=1}^{\infty} \frac{n}{n+1}$ diverges. ◀

Divergence Test

Theorem: [Divergence Test]

Assume that $\sum_{n=1}^{\infty} a_n$ converges, then

$$\lim_{n\to\infty} a_n = 0.$$

In particular, if $\lim_{n\to\infty} a_n$ does not exist or

$$\lim_{n\to\infty} a_n \neq 0,$$

then $\sum_{n=1}^{\infty} a_n$ diverges.

Question: Does the converse hold? That is, if $\lim_{k\to\infty} a_k = 0$, does $\sum_{n=1}^{\infty} a_n$ converge?

Divergence Test

Example: Consider $\sum_{n=1}^{\infty} \frac{1}{n}$.

$$
\begin{aligned}
S_1 &= 1 = \frac{2}{2} \\
S_2 &= 1 + \frac{1}{2} = \frac{3}{2} \\
S_4 &= 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} \\
&\geq 1 + \frac{1}{2} + \left(\cancelto{\frac{1}{4}}{\frac{1}{3}} + \frac{1}{4}\right) \\
&= \frac{4}{2}
\end{aligned}
$$

$$
\begin{aligned}
S_8 &= S_4 + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} \\
&\geq \cancelto{\frac{4}{2}}{S_4} + \cancelto{\frac{1}{8}}{\frac{1}{5}} + \cancelto{\frac{1}{8}}{\frac{1}{6}} + \cancelto{\frac{1}{8}}{\frac{1}{7}} + \frac{1}{8} \\
&= \frac{5}{2}
\end{aligned}
$$

Observation: For each $j \in \mathbb{N}$

$$S_{2^j} \geq \frac{j+2}{2} \to \infty.$$

Divergence Test

We have shown that for $\sum_{n=1}^{\infty} \frac{1}{n}$, the sequence $\{S_k\}$ of partial sums is unbounded and therefore divergent. Hence,

$$\sum_{n=1}^{\infty} \frac{1}{n} = \infty$$

even though

$$\lim_{n\to\infty} \frac{1}{n} = 0.$$

Definition: [Harmonic Series]

The divergent series

$$\sum_{n=1}^{\infty} \frac{1}{n}$$

is called the *Harmonic Series*.

Harmonic Series

Problem: How long would it take your computer to add up enough terms in the series

$$\sum_{n=1}^{\infty} \frac{1}{n}$$

so that

$$S_k > 100?$$