The Comparison Test for Series

Created by

Barbara Forrest and Brian Forrest
Example

**Problem:** Does $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converge or diverge?

**Key Observation:** We saw that

$$\sum_{n=2}^{\infty} \frac{1}{n^2 - n} = 1$$

and that if $n \geq 2$, then

$$\frac{1}{n^2} < \frac{1}{n^2 - n}.$$ 

If

$$T_k = \sum_{n=1}^{k} \frac{1}{n^2} \quad \text{and} \quad S_k = \sum_{n=2}^{k} \frac{1}{n^2 - n}$$

then for $k \geq 2$,

$$T_k = 1 + \sum_{n=2}^{k} \frac{1}{n^2} \leq 1 + \sum_{n=2}^{k} \frac{1}{n^2 - n} < 2$$

Hence $\{T_k\}$ is bounded above by 2 and $\{T_k\}$ converges, as does $\sum_{n=1}^{\infty} \frac{1}{n^2}$.
Comparison Test

**Question:** If $0 \leq a_n \leq b_n$ and $\sum_{n=1}^{\infty} b_n$ converges, must $\sum_{n=1}^{\infty} a_n$ converge?

**Theorem: [The Comparison Test for Series]**

Assume that $0 \leq a_n \leq b_n$ for each $n \in \mathbb{N}$.

1) If $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.

2) If $\sum_{n=1}^{\infty} a_n$ diverges, then $\sum_{n=1}^{\infty} b_n$ diverges.
Comparison Test

Proof of the Comparison Test:
Assume that $0 \leq a_n \leq b_n$ for each $n \in \mathbb{N}$. Let

$$T_k = \sum_{n=1}^{k} a_n \quad \text{and} \quad S_k = \sum_{n=1}^{k} b_n$$

1) Assume that $\sum_{n=1}^{\infty} b_n$ converges with $\sum_{n=1}^{\infty} b_n = S$. Then

$$T_k = \sum_{n=1}^{k} a_n \leq S_k = \sum_{n=1}^{k} b_n \leq S$$

for each $k \in \mathbb{N}$. It follows that $\{T_k\}$ is bounded and that

$$\sum_{n=1}^{\infty} a_n$$

also converges.
Proof of the Comparison Test:
Assume that $0 \leq a_n \leq b_n$ for each $n \in \mathbb{N}$. Let

$$T_k = \sum_{n=1}^{k} a_n \quad \text{and} \quad S_k = \sum_{n=1}^{k} b_n$$

2) Assume that $\sum_{n=1}^{\infty} a_n$ diverges. Let $M > 0$. Then we can find an $N \in \mathbb{N}$ so that $M \leq T_N$. But if $k \geq N$,

$$M \leq T_N \leq S_N \leq S_k$$

and hence

$$\sum_{n=1}^{\infty} b_n$$

also diverges.
Comparison Test

Three Important Observations:

1) If $\sum_{n=1}^{\infty} a_n$ converges, then we cannot say anything about $\sum_{n=1}^{\infty} b_n$.

2) If $\sum_{n=1}^{\infty} b_n$ diverges, then we cannot say anything about $\sum_{n=1}^{\infty} a_n$.

3) Since the first few terms do not affect whether or not a series diverges, for the Comparison Test to hold, we really only need that

$$0 \leq a_n \leq b_n$$

for each $n \geq K$, where $K \in \mathbb{N}$. That is, the conditions of the theorem need only be satisfied by the elements of the tails of the two sequences.
**Examples**

**Example:** We have seen that the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges and that

$$0 < \frac{1}{n^3} \leq \frac{1}{n^2}$$

for all $n \in \mathbb{N}$.

If $a_n = \frac{1}{n^3}$ and $b_n = \frac{1}{n^2}$, then the Comparison Test shows that the series

$$\sum_{n=1}^{\infty} \frac{1}{n^3}$$

also converges.

In fact, if $p \geq 2$, then

$$0 < \frac{1}{n^p} \leq \frac{1}{n^2}$$

so

$$\sum_{n=1}^{\infty} \frac{1}{n^p}$$

converges.
Examples

Note: We know that $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges and that

$$0 < \frac{1}{n} \leq \frac{1}{\sqrt{n}}$$

for all $n \in \mathbb{N}$.

If $a_n = \frac{1}{n}$ and $b_n = \frac{1}{\sqrt{n}}$, then the Comparison Test shows that the series

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$$

also diverges.

In fact, if $p \leq 1$, then

$$0 < \frac{1}{n} \leq \frac{1}{n^p}$$

so

$$\sum_{n=1}^{\infty} \frac{1}{n^p}$$

diverges.
Examples

**Question:** We know that the series \( \sum_{n=1}^{\infty} \frac{1}{n^p} \) diverges if \( p \leq 1 \) and it converges if \( p \geq 2 \). What happens if

\[
1 < p < 2
\]

**Note:** If \( 1 < p < 2 \), then

\[
\frac{1}{n^2} \leq \frac{1}{n^p} \leq \frac{1}{n}
\]

so the Comparison Test fails!
**Example:** Show that the series

\[
\sum_{n=1}^{\infty} \sin \left( \frac{1}{n^2} \right)
\]

converges.

We know that \(0 < \sin(x) < x\) for all \(x \in (0, 1)\) so

\[
0 < \sin \left( \frac{1}{n^2} \right) < \frac{1}{n^2}
\]

for all \(n \in \mathbb{N}\).

If \(a_n = \sin \left( \frac{1}{n^2} \right)\) and \(b_n = \frac{1}{n^2}\), then the Comparison Test shows that the series

\[
\sum_{n=1}^{\infty} \sin \left( \frac{1}{n^2} \right)
\]

also converges.
**Examples**

**Question:** Does the series

\[ \sum_{n=1}^{\infty} \sin \left( \frac{1}{n} \right) \]

converge?

We know that

\[ 0 < \sin \left( \frac{1}{n} \right) < \frac{1}{n} \]

for all \( n \in \mathbb{N} \) but \( \sum_{n=1}^{\infty} \frac{1}{n} \) diverges so the Comparison Test fails.

Since \( \lim_{n \to \infty} \frac{1}{n} = 0 \), the Fundamental Trig Limits shows that

\[ \lim_{n \to \infty} \frac{\sin \left( \frac{1}{n} \right)}{\frac{1}{n}} = 1 \]

so for large \( n \) we have

\[ \sin \left( \frac{1}{n} \right) \approx \frac{1}{n} \]

Does this mean that

\[ \sum_{n=1}^{\infty} \sin \left( \frac{1}{n} \right) \]

also diverges?