Arithmetic of Series

Created by

Barbara Forrest and Brian Forrest

Arithmetic of Series

Question: Assume that $\sum_{n=1}^{\infty} a_n$ converges, what can we say about the series $\sum_{n=1}^{\infty} 2 \cdot a_n$?

If

$$S_k = \sum_{n=1}^{k} a_n \quad \text{and} \quad T_k = \sum_{n=1}^{k} 2 \cdot a_n,$$

then

$$T_k = \sum_{n=1}^{k} 2 \cdot a_n = 2 \cdot \sum_{n=1}^{k} a_n = 2 \cdot S_k$$

Since $\sum_{n=1}^{\infty} a_n$ converges, $\lim_{k\to\infty} S_k$ exists. Let

$$\lim_{k\to\infty} S_k = \sum_{n=1}^{\infty} a_n = S$$

Then

$$\lim_{k\to\infty} T_k = \lim_{k\to\infty} 2 \cdot S_k = 2 \cdot \lim_{k\to\infty} S_k = 2 \cdot S$$

Therefore, $\sum_{n=1}^{\infty} 2 \cdot a_n$ also converges and

$$\sum_{n=1}^{\infty} 2 \cdot a_n = 2 \cdot \sum_{n=1}^{\infty} a_n$$

Arithmetic of Series

Theorem: [Arithmetic Rules for Series I]

Assume that $\sum\limits_{n=1}^{\infty} a_n$ and $\sum\limits_{n=1}^{\infty} b_n$ both converge.

1. The series $\sum\limits_{n=1}^{\infty} ca_n$ converges for every $c \in \mathbb{R}$ and

$$\sum_{n=1}^{\infty} ca_n = c \sum_{n=1}^{\infty} a_n.$$

2. The series $\sum\limits_{n=1}^{\infty} (a_n + b_n)$ converges and

$$\sum_{n=1}^{\infty} (a_n + b_n) = \sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} b_n.$$

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Remark: Given a series $\sum_{n=1}^{\infty} a_n$ and $j \in \mathbb{N}$, let

$$\sum_{n=j}^{\infty} a_n = a_j + a_{j+1} + a_{j+2} + a_{j+3} + \cdots .$$

We say that $\sum_{n=j}^{\infty} a_n$ *converges* if

$$\lim_{k\to\infty} T_k$$

exists, where

$$T_k = \sum_{n=j}^{j+k-1} a_n = a_j + a_{j+1} + a_{j+2} + a_{j+3} + \cdots + a_{j+k-1}.$$

Arithmetic of Series

Theorem: [Arithmetic Rules for Series II]

1. If $\sum_{n=1}^{\infty} a_n$ converges, then $\sum_{n=j}^{\infty} a_n$ also converges for **each** j.
2. If $\sum_{n=j}^{\infty} a_n$ converges for **some** j, then $\sum_{n=1}^{\infty} a_n$ converges.

In either of the above two cases,

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + \cdots + a_{j-1} + \sum_{n=j}^{\infty} a_n$$

Important Observation: If we have a sequence

$$a_1, a_2, a_3, \cdots, a_n, \cdots$$

and we change the first $j - 1$ terms to make a new sequence

$$b_1, b_2, b_3, \cdots, b_n, \cdots$$

where $b_n = a_n$ and if $n \geq j$. Then the series $\sum_{n=j}^{\infty} a_n$ and $\sum_{n=j}^{\infty} b_n$ are identical.

Hence **either both** $\sum_{n=1}^{\infty} a_n$ **and** $\sum_{n=1}^{\infty} b_n$ **converge or they both diverge!!!**

Bouncing Ball

Problem: A ball is launched straight up from the ground to a height of 30m. When the ball returns to the ground it will bounce to a height that is exactly $\frac{1}{3}$ of its previous height. Assuming that the ball continues to bounce each time it returns to the ground, how far does the ball travel before coming to rest?

Solution:

1. Prior to returning to the ground for the first time, the ball travels 30m on its way up and then 30m down for a total of $2(30) = 60m$.
2. On the first bounce, the ball will travel upwards $\frac{30}{3}$m and down again the same distance for a total of $2(\frac{30}{3})m$.
3. On the second bounce, the ball will again travel upwards one third the distance it traveled on the first bounce or

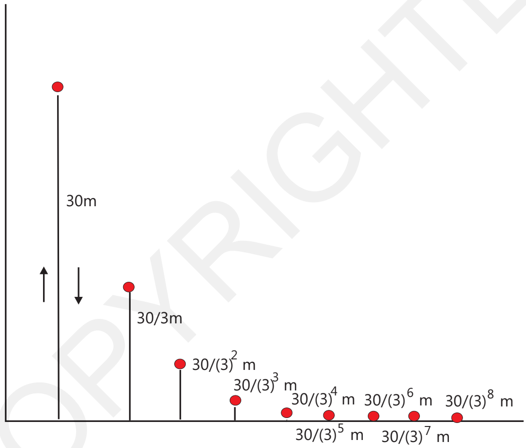
$$\frac{1}{3}(\frac{30}{3}) = (\frac{30}{3^2})m.$$

   With the downward trip, the second bounce covers a distance of $2\frac{30}{3^2}$m.

Bouncing Ball

4. On the n-th bounce, the ball will travel a distance of $\mathbf{2(\frac{30}{3^n})}m$. The total distance D the ball travels will be the sum of each of these distances.

Bouncing Ball



Bouncing Ball

4. On the n-th bounce, the ball will travel a distance of $2(\frac{30}{3^n})m$. The total distance D the ball travels will be the sum of each of these distances.

Therefore,

$$\begin{aligned} D &= 2(30) + 2(\frac{30}{3}) + 2(\frac{30}{3^2}) + 2(\frac{30}{3^3}) + \cdots + 2(\frac{30}{3^n}) + \cdots \\ &= 2(30)[1 + \frac{1}{3} + \frac{1}{3^2} + \frac{1}{3^3} + \cdots + \frac{1}{3^n} + \cdots] \\ &= 60 \sum_{n=0}^{\infty} (\frac{1}{3})^n \\ &= \frac{60}{1 - \frac{1}{3}} \\ &= 90 \text{ meters.} \end{aligned}$$

Question: How long does it take the ball to return to rest?