Absolute vs Conditional Convergence

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Absolute Convergence

Important Observation:

1) The Harmonic Series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges, while the Alternating Series $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n}$ converges even though the terms have the same order of magnitude,

$$\left| \frac{1}{n} \right| = \frac{1}{n} = \left| (-1)^{n-1} \frac{1}{n} \right|$$

2) On the other hand, both $\sum_{n=1}^{\infty} \frac{1}{n^2}$ and $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n^2}$ converge because $\frac{1}{n^2}$ is *small enough*!

Question 1: Is there a way to detect if a series will converge because its terms are *small enough*?

Absolute Convergence

Question 2: Aside from the AST all of our tests apply to positive series. Is there any additional method for determining if an arbitrary series $\sum_{n=1}^{\infty} a_n$ converges?

Definition: [Absolute Convergence]

We say that a series $\sum_{n=1}^{\infty} a_n$ *converges absolutely* or is *absolutely convergent* if the series

$$\sum_{n=1}^{\infty} |\, a_n \,|$$

converges.

Question 3: If $\sum_{n=1}^{\infty} a_n$ converges absolutely, does it converge?

Absolute Convergence

Theorem: [Absolute Convergence Theorem]

Assume that the series $\sum_{n=1}^{\infty} a_n$ converges absolutely. Then $\sum_{n=1}^{\infty} a_n$ converges.

Proof: Assume that

$$\sum_{n=1}^{\infty} | a_n |$$

converges. Then so does $\sum_{n=1}^{\infty} 2 | a_n |$. Let

$$b_n = a_n + | a_n | \Rightarrow 0 \leq b_n \leq 2 | a_n | .$$

By the Comparison Test $\sum_{n=1}^{\infty} b_n$ converges.

Since $a_n = b_n - |a_n|$, it follows that $\sum_{n=1}^{\infty} a_n$ also converges and

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} (a_n + | a_n |) - \sum_{n=1}^{\infty} | a_n |$$

Example:

Example 1: Let $\{a_n\} = \{1, \frac{1}{2^2}, \frac{-1}{3^2}, \frac{1}{4^2}, \frac{1}{5^2}, \frac{-1}{6^2}, \cdots\}$. Then

$$| a_n | = \frac{1}{n^2}$$

so $\sum_{n=1}^{\infty} | a_n |$ converges and the series $\sum_{n=1}^{\infty} a_n$ converges absolutely.

Example:

Example 2: Let $\{b_n\} = \{(-1)^{n+1}\sin(\frac{1}{\sqrt{n}})\}$.
Since

1) $\sin(\frac{1}{\sqrt{n}}) > 0$ for all $n \in \mathbb{N}$,
2) $\sin(\frac{1}{\sqrt{n+1}}) < \sin(\frac{1}{\sqrt{n}})$ for all $n \in \mathbb{N}$,
3) $\lim\limits_{n\to\infty} \sin(\frac{1}{\sqrt{n}}) = 0$,

the series $\sum\limits_{n=1}^{\infty} (-1)^{n+1}\sin(\frac{1}{\sqrt{n}})$ converges by the AST.

However, since the FTL shows that

$$\lim_{n\to\infty} \frac{\sin(\frac{1}{\sqrt{n}})}{\frac{1}{\sqrt{n}}} = 1,$$

and $\sum\limits_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ diverges, $\sum\limits_{n=1}^{\infty} |(-1)^{n+1}\sin(\frac{1}{\sqrt{n}})| = \sum\limits_{n=1}^{\infty} \sin(\frac{1}{\sqrt{n}})$ diverges.
Hence $\sum\limits_{n=1}^{\infty} (-1)^{n+1}\sin(\frac{1}{\sqrt{n}})$ is not absolutely convergent.

Conditional Converge:

Definition: [Conditional Convergence]

A series $\sum_{n=1}^{\infty} a_n$ is said to be *conditionally convergent* if it converges, but it is not absolutely convergent.

Question 4: Why do we care if a series converges absolutely rather than conditionally?

Rearrangement of Series :

Question 5: For a finite sum *order does not matter*. That is

$$a + b + c + d = d + c + b + a$$

Is this true for infinite sums?

Definition: [Rearrangement of a Series] Given a sequence $\{a_n\}$ and a $1-1$ and onto function $\phi : \mathbb{N} \to \mathbb{N}$, if we let

$$b_n = a_{\phi(n)},$$

then the series $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} a_{\phi(n)}$ is called a *rearrangement* of the series $\sum_{n=1}^{\infty} a_n$.

Rearrangement of Series :

Question 6: What do we know about the convergence of $\sum_{n=1}^{\infty} a_{\phi(n)}$ relative to that of $\sum_{n=1}^{\infty} a_n$. In particular, if $\sum_{n=1}^{\infty} a_n$ converges, does $\sum_{n=1}^{\infty} a_{\phi(n)}$ also converge with

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} a_{\phi(n)}?$$

Rearrangement of Series :

Facts:

1) If $\sum_{n=1}^{\infty} a_n$ converges absolutely, then $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} a_{\phi(n)}$ for every rearrangement $\sum_{n=1}^{\infty} a_{\phi(n)}$ of $\sum_{n=1}^{\infty} a_n$.

2) If $\sum_{n=1}^{\infty} a_n$ converges conditionally, then $\sum_{n=1}^{\infty} a_{\phi(n)}$ may or may not converge.

3) If $\sum_{n=1}^{\infty} a_n$ converges conditionally and if $\alpha \in \mathbb{R} \cup \{-\infty, \infty\}$, then there exists a $1-1$ and onto function $\phi : \mathbb{N} \to \mathbb{N}$ such that

$$\sum_{n=1}^{\infty} a_{\phi(n)} = \alpha.$$

Summary: Absolutely convergent series are stable, conditionally convergent series are not!