Alternating Series Test Part II: The AST and Error Estimation

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Alternating Series

Recall: If $a_n = \frac{1}{n}$, then
i) $a_n > 0$
ii) $a_{n+1} < a_n$
iii) $\lim_{n \to \infty} a_n = 0$

Using these properties we showed that the Alternating Series

$$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \cdots + (-1)^{n+1} \frac{1}{n} + \cdots$$

converged. Moreover, if

$$S_k = \sum_{n=1}^{k} (-1)^{n+1} \frac{1}{n} \quad \text{and} \quad S = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n},$$

then

$$|S - S_k| < a_{k+1} = \frac{1}{k + 1}$$
Theorem: [Alternate Series Test]

1) \( a_n > 0 \) for all \( n \in \mathbb{N} \),

2) \( a_{n+1} < a_n \) for all \( n \in \mathbb{N} \),

3) \( \lim_{n \to \infty} a_n = 0 \).

Then the series

\[
\sum_{n=1}^{\infty} (-1)^{n+1} a_n
\]

converges.

Moreover, if \( S_k = \sum_{n=1}^{k} (-1)^{n+1} a_n \) and \( S = \sum_{n=1}^{\infty} (-1)^{n+1} a_n \), then

\[
| S_k - S | < a_{k+1}.
\]
We get

1) \( \{S_{2k-1}\} \) is decreasing and bounded below by 0
\[ \Rightarrow \{S_{2k-1}\} \rightarrow L. \]

2) \( \{S_{2k}\} \) is increasing and bounded above by \( a_1 \)
\[ \Rightarrow \{S_{2k}\} \rightarrow M. \]

3) \[ \lim_{k \rightarrow \infty} |S_{2k-1} - S_{2k}| = \lim_{k \rightarrow \infty} a_{2k} = 0 \Rightarrow L = M = S. \]

Finally, since \( S \) is between \( S_k \) and \( S_{k+1} \) for each \( k \),

\[ |S_k - S| < |S_k - S_{k+1}| = a_{k+1}. \]
Example: Show that \( \sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n^5} \) converges to some \( S \) and that

\[
\left| \sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n^5} - \frac{31}{32} \right| < \frac{1}{100}.
\]

Solution: With \( a_n = \frac{1}{n^5} \) it is clear that

1) \( a_n > 0 \) for all \( n \in \mathbb{N} \),

2) \( a_{n+1} < a_n \) for all \( n \in \mathbb{N} \),

3) \( \lim_{n \to \infty} a_n = 0 \).

so the series converges by the AST.

We know that \( \frac{31}{32} = 1 - \frac{1}{2^5} = \sum_{n=1}^{2} (-1)^{n+1} \frac{1}{n^5} = S_2 \). Hence

\[
\left| \sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n^5} - \frac{31}{32} \right| = \left| S - S_2 \right| < a_3 = \frac{1}{3^5} = \frac{1}{243} < \frac{1}{100}.
\]
Example: The Integral Test shows that the series
\[ \sum_{n=2}^{\infty} \frac{1}{n(\ln(n))^2} \]
converges and the AST shows that
\[ \sum_{n=2}^{\infty} (-1)^n \frac{1}{n(\ln(n))^2} \]
converges.
Let
\[ T_k = \sum_{n=2}^{k} \frac{1}{n(\ln(n))^2} \quad \text{and} \quad T = \sum_{n=2}^{\infty} \frac{1}{n(\ln(n))^2} \]
and let
\[ S_k = \sum_{n=2}^{k} (-1)^n \frac{1}{n(\ln(n))^2} \quad \text{and} \quad S = \sum_{n=2}^{\infty} (-1)^n \frac{1}{n(\ln(n))^2} \]
How large must \( k \) be so that
\[ |T - T_k| < \frac{1}{100} \quad \text{and} \quad |S - S_k| < \frac{1}{100} \]
respectively?
Example:

Solution: The Integral Test shows that

\[ |T - T_k| < \int_k^\infty \frac{1}{x(ln(x))^2} \, dx = \frac{1}{\ln(k)} \]

We want

\[ \frac{1}{\ln(k)} < \frac{1}{100} \Rightarrow k > e^{100} \approx 10^{43} \]

The AST shows that

\[ |S - S_k| < \frac{1}{(k + 1)(ln(k + 1))^2} \]

We want

\[ \frac{1}{(k + 1)(ln(k + 1))^2} < \frac{1}{100} \Rightarrow k \geq 14 \]