Graphical and Numerical Solutions of DE’s

Created by

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**Graphical and Numerical Solutions of DE’s**

**Remark:** Most DE’s can not be solved explicitly but we can use linear approximation to help us build both *graphical* and *numerical* approximations to solutions.

**Recall:** If $f$ is differentiable at $x_0$, then $f$ can be approximated near $x_0$ by its linear approximation

$$L_{x_0}(x) = f(x_0) + f'(x_0)(x - x_0)$$

and the graph of $L_{x_0}$ is the tangent line to the graph of $f$ through the point $(x_0, f(x_0))$. 
Direction Fields

Graphical Solution: The Strategy

Given a differential equation

\[ y' = f(x, y) \]

if there is a solution \( \phi \) whose graph includes the point \((x_0, y_0)\), then the slope of the tangent line to the graph of \( \phi \) through \((x_0, y_0)\) is

\[ m = f(x_0, y_0) \]

We can construct a local graphical approximation to the function \( \phi \) near \((x_0, y_0)\) by looking at a short segment of this tangent line.

A collection of such segments is called a direction field for the differential equation.
**Direction Fields**

| $x$ | $y$ | tangent line slope from DE  
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<tbody>
<tr>
<td>-2</td>
<td>0</td>
<td>$y' = -2 + 0 = -2$</td>
</tr>
<tr>
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<td>0</td>
<td>$y' = -1 + 0 = -1$</td>
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<td>$y' = 0 + 0 = 0$</td>
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<td>$y' = 1 + 0 = 1$</td>
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<td>1</td>
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<td>$y' = 1 + 1 = 2$</td>
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<td>$y' = 2 + 1 = 3$</td>
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<tr>
<td>-1</td>
<td>-1</td>
<td>$y' = -1 + -1 = -2$</td>
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**Example:** Consider the linear DE  

$$y' = x + y.$$
Direction Field for \( y' = x + y \)
Direction Fields

Note: \( y = -x - 1 \) is a solution to \( y' = x + y \) with \( y(-1) = 0 \).

Solution Curve for \( y(0) = 1 \)
Note: $y = -x - 1$ is a solution to $y' = x + y$ with $y(-1) = 0$. 
**Euler’s Method**: This is an algorithm for building a numerical approximation on a closed interval \([a, b]\) to a solution to an initial value problem

\[
y' = f(x, y)
\]

with \(y(a) = y_0\).

**Step 1**: Determine a partition

\[
P = \{a = x_0 < x_1 < x_2 < \cdots < x_n = b\}
\]

of \([a, b]\)
Euler’s Method

Step 2: Assume that $\phi$ is a solution to $y' = f(x, y)$ with $\phi(x_0) = y_0$. Then

$$L_{x_0}(x) = y_0 + f(x_0, y_0)(x - x_0)$$

so we can assume that

$$\phi(x) = y_0 + f(x_0, y_0)(x - x_0)$$
on $[x_0, x_1]$. 
**Euler’s Method**

**Step 3:** Let

\[ y_1 = L_{x_0}(x_1) = y_0 + f(x_0, y_0)(x_1 - x_0) \]

and define

\[ \phi(x) = L_{x_1}(x) = y_1 + f(x_1, y_1)(x - x_1) \]

on \([x_1, x_2]\).
Euler’s Method

Step 3: Let

\[ y_2 = L_{x_1}(x_2) = y_1 + f(x_1, y_1)(x_2 - x_1) \]

and define

\[ \phi(x) = L_{x_2}(x) = y_2 + f(x_2, y_2)(x - x_2) \]

on \([x_2, x_3]\) and continue until you reach \(x_n = b\).