Volumes of Revolution: Shell Method

Created by

Barbara Forrest and Brian Forrest
**Volumes by the Shell Method**

**Problem:** Assume that \( f \) and \( g \) are continuous on \([a, b]\), with \( a \geq 0 \) and \( f(x) \leq g(x) \) on \([a, b]\).

Let \( W \) be the region bounded by the graphs of \( f \) and \( g \) and the lines \( x = a \) and \( x = b \).

Find the volume \( V \) of the solid obtained by rotating the region \( W \) around the \( y \)-axis.
Volumes by the Shell Method

Construct a regular n-partition of \([a, b]\) with

\[a = x_0 < x_1 < \cdots < x_{i-1} < x_i < \cdots < x_n = b.\]

This partition subdivides the region \(W\) into \(n\) subregions.

Let \(W_i\) denote the subregion of \(W\) on the interval \([x_{i-1}, x_i]\)

Let \(V_i\) be the volume obtained by rotating \(W_i\) around the \(y\)-axis so that

\[V = \sum_{i=1}^{n} V_i.\]
Volumes by the Shell Method

Approximate $W_i$ by the rectangle $R_i$ with height $g(x_i) - f(x_i)$, and base on the line $y = f(x_i)$ and top on the line $y = g(x_i)$ in the interval $[x_{i-1}, x_i]$. 
Volumes by the Shell Method

If $\Delta x_i$ is small, then $V_i$ is approximately equal to the volume obtained by rotating $R_i$ around the $y$-axis.

Rotating $R_i$ generates a thin cylindrical shell $S_i$.

For this reason, this method for finding volumes is called the Shell Method (or Cylindrical Shell Method).
Volumes by the Shell Method

The volume $V_i^*$ of the shell generated by $R_i$ is

$$(\text{circumference}) \times (\text{height}) \times (\text{thickness})$$

which is the same as

$$2\pi \times (\text{radius}) \times (\text{height}) \times (\text{thickness}).$$

The height of the shell is $g(x_i) - f(x_i)$, its thickness is $\Delta x_i$, and the radius of revolution is $x_i$ (the distance from the $y$-axis). Therefore, the volume $V_i^*$ of $S_i$ is

$$2\pi x_i (g(x_i) - f(x_i)) \Delta x_i.$$
Volumes by the Shell Method

\[
\text{thickess} = \Delta x_i \\
\text{radius} = x_i \\
\text{circumference} = 2\pi x_i \\
\text{height} = g(x_i) - f(x_i)
\]

\[
\text{Volume} = 2\pi x_i (g(x_i) - f(x_i)) \Delta x_i
\]
Volumes by the Shell Method

\[ V = \sum_{i=1}^{n} V_i \]

\[ \approx \sum_{i=1}^{n} V_i^* \]

\[ = \sum_{i=1}^{n} 2\pi x_i (g(x_i) - f(x_i)) \Delta x_i \]

Let \( n \to \infty \) to get

\[ V = \int_{a}^{b} 2\pi x(g(x) - f(x)) \, dx. \]
Volumes of Revolution: The Shell Method

Let \( a \geq 0 \). Let \( f \) and \( g \) be continuous on \([a, b] \) with \( f(x) \leq g(x) \) for all \( x \in [a, b] \).

Let \( W \) be the region bounded by the graphs of \( f \) and \( g \), and the lines \( x = a \) and \( x = b \).

Then the volume \( V \) of the solid of revolution obtained by rotating the region \( W \) around the \( y \)-axis is given by

\[
V = \int_a^b 2\pi x(g(x) - f(x)) \, dx.
\]
Volumes by the Shell Method

Example:

Find the volume of the solid obtained by revolving the closed region in the first quadrant bounded by the graphs of \( g(x) = x \) and \( f(x) = x^2 \) around the \( y \)-axis.

You should verify that the graphs intersect in the first quadrant when \( x = 0 \) and \( x = 1 \) on the interval \([0, 1]\) with \( f(x) \leq g(x) \).
Volumes by the Shell Method

Example (continued):

\[ y \]

Graphical representation of functions:

\[ f(x) = x^2 \]
\[ g(x) = x \]

Graph showing the region of integration.

\[ V = \int_{0}^{1} 2\pi x (g(x) - f(x)) \, dx \]

\[ = \int_{0}^{1} 2\pi x (x - x^2) \, dx \]

\[ = \int_{0}^{1} 2\pi (x^2 - x^3) \, dx = \frac{\pi}{6} \]