Areas Between Curves: Examples

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Recall:

Area Between Curves

Let $f$ and $g$ be continuous on $[a, b]$. Let $A$ be the region bounded by the graphs of $f$ and $g$, the line $t = a$ and the line $t = b$. Then the area of region $A$ is given by

$$A = \int_{a}^{b} |g(t) - f(t)| \, dt.$$
Example:

Find the area $A$ of the closed region bounded by the graphs of the functions $g(x) = x^2$ and $f(x) = x^3$.

This area is the shaded region in the diagram.
Example (continued):

The graphs cross when \( x^3 = x^2 \) or when

\[
0 = x^3 - x^2
\]

\[
\Rightarrow 0 = x^2(x - 1)
\]

This occurs when \( x = 0 \) and \( x = 1 \).

The area is bounded by the functions \( g(x) = x^2 \) and \( f(x) = x^3 \) between the lines \( x = 0 \) and \( x = 1 \).
Example (continued):

Notice that \( x^2 \geq x^3 \) on the interval \([0, 1]\).

Then the area is

\[
A = \int_0^1 (x^2 - x^3) \, dx
\]

\[
= \left( \frac{x^3}{3} - \frac{x^4}{4} \right) \bigg|_0^1
\]

\[
= \left( \frac{1}{3} - \frac{1}{4} \right) - (0 - 0)
\]

\[
= \frac{1}{12}
\]
Example:

Find the total area $A$ of the closed regions bounded by the graphs of the functions $f(x) = x$ and $g(x) = x^3$.

The shaded regions in the diagram represent $A$. 
Example (continued):

Points of Intersection

The graphs intersect where $x^3 = x$.

\[
0 = x^3 - x \\
\Rightarrow 0 = x(x^2 - 1) \\
\Rightarrow 0 = x(x + 1)(x - 1)
\]

The points of intersection occur at $x = -1$, $x = 0$, and $x = 1$. 
Example (continued):

We **cannot** apply the Fundamental Theorem of Calculus directly to $|x^3 - x|$ to calculate the area since $f$ and $g$ intersect on the interval $[-1, 1]$. 
Example (continued):

Instead, we must consider the area in two parts, $A_1$ and $A_2$. 

\[
f(x) = x \quad \text{and} \quad g(x) = x^3
\]
Example (continued):

Case: Area of A1

On the interval $[-1, 0]$

$x^3 \geq x$

so

$$\int_{-1}^{0} |x^3 - x| \, dx = \int_{-1}^{0} (x^3 - x) \, dx$$

This integral represents $A_1$, the shaded area in the diagram.
Example (continued):

Case: Area of $A_2$

On the interval $[0, 1]$

$$x \geq x^3$$

so

$$\int_{0}^{1} |x^3 - x| \, dx = \int_{0}^{1} (x - x^3) \, dx$$

This integral represents $A_2$, the shaded area in the diagram.
Example (continued):

Total Area Between the Curves

The total area $A$ between the curves $f(x) = x$ and $g(x) = x^3$ on the interval $[-1, 1]$ is

$$A = A_1 + A_2.$$
Example (continued):

Total Area Between the Curves

\[
A = \int_{-1}^{1} |x^3 - x| \, dx
\]

\[
= A_1 + A_2
\]

\[
= \int_{-1}^{0} |x^3 - x| \, dx + \int_{0}^{1} |x^3 - x| \, dx
\]

\[
= \int_{-1}^{0} (x^3 - x) \, dx + \int_{0}^{1} (x - x^3) \, dx
\]

\[
= \left( \frac{x^4}{4} - \frac{x^2}{2} \right) \bigg|_{-1}^{0} + \left( \frac{x^2}{2} - \frac{x^4}{4} \right) \bigg|_{0}^{1}
\]

\[
= \left( (0 - 0) - \left( \frac{1}{4} - \frac{1}{2} \right) \right) + \left( \left( \frac{1}{2} - \frac{1}{4} \right) - (0 - 0) \right)
\]

\[
= \frac{1}{4} + \frac{1}{4}
\]

\[
= \frac{1}{2}
\]