Examples of Integration by Parts

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Example: Evaluate $\int e^x \sin(x)\, dx$.

Strategy: Nothing can be eliminated by differentiation but we can take advantage of the cyclic nature of $\sin(x)$ and $\cos(x)$ with respect to differentiation.

Let $f(x) = \sin(x)$, $g'(x) = e^x$. Then $f'(x) = \cos(x)$ and $g(x) = e^x$ gives

$$\int e^x \sin(x)\, dx = e^x \sin(x) - \int e^x \cos(x)\, dx$$

Apply Integration by Parts again with $f(x) = \cos(x)$ and $g'(x) = e^x$ to get

$$\begin{aligned}
\int e^x \sin(x)\, dx &= e^x \sin(x) - \int e^x \cos(x)\, dx \\
&= e^x \sin(x) - \left[e^x \cos(x) + \int e^x \sin(x)\, dx\right] \\
&= e^x \sin(x) - e^x \cos(x) - \int e^x \sin(x)\, dx
\end{aligned}$$

Hence

$$2 \int e^x \sin(x)\, dx = e^x \sin(x) - e^x \cos(x)$$

and

$$\int e^x \sin(x)\, dx = \frac{e^x \sin(x) - e^x \cos(x)}{2} + C$$

More Examples

Example: Evaluate $\int \ln(x)\,dx = \int 1 \cdot \ln(x)\,dx$

Note: There is an "invisible" second factor in the integrand: $1 \cdot \ln(x)$

Let $g'(x) = 1$ and $f(x) = \ln(x)$. Then $f'(x) = \frac{1}{x}$ and $g(x) = x$.

$$\begin{aligned}
\int \ln(x)\,dx = \int f(x)g'(x)\,dx &= f(x)g(x) - \int f'(x)g(x)\,dx \\
&= x\ln(x) - \int \frac{1}{x} \cdot x\,dx \\
&= x\ln(x) - \int 1\,dx \\
&= x\ln(x) - x + C
\end{aligned}$$

More Examples

Example: Evaluate $\int \arctan(x)\,dx = \int 1 \cdot \arctan(x)\,dx$.

Let $f(x) = \arctan(x)$ and $g'(x) = 1$. Then $f'(x) = \frac{1}{1+x^2}$ and $g(x) = x$. We obtain

$$\int \arctan(x)\,dx = \int f(x)g'(x)\,dx = f(x)g(x) - \int f'(x)g(x)\,dx$$
$$= x\arctan(x) - \int \frac{x}{1+x^2}\,dx$$

Making the substitution $u = 1 + x^2 \Rightarrow du = 2x\,dx$ we now have

$$\int \frac{x}{1+x^2}\,dx = \frac{1}{2}\int \frac{2x}{1+x^2}\,dx$$
$$= \frac{1}{2}\int \frac{1}{u}\,du$$
$$= \frac{1}{2}\ln(|u|) + C_0$$
$$= \frac{1}{2}\ln(|1+x^2|) + C_0$$

More Examples

Example: Evaluate $\int \arctan(x)\, dx = \int 1 \cdot \arctan(x)\, dx$.

Therefore

$$\begin{aligned}
\int \arctan(x)\, dx &= x \arctan(x) - \int \frac{x}{1+x^2}\, dx \\
&= x \arctan(x) - \left(\frac{1}{2} \ln(|1+x^2|) + C_0\right) \\
&= x \arctan(x) - \frac{1}{2} \ln(|1+x^2|) + C
\end{aligned}$$

More Examples

Remark: The FTC implies the following:

Theorem: [Integration by Parts]

Assume that $f'(x)$ and $g'(x)$ are both continuous on $[a, b]$. Then

$$\int_a^b f(x)g'(x)\,dx = f(x)g(x)\mid_a^b - \int_a^b f'(x)g(x)\,dx$$