Introduction to Improper Integrals

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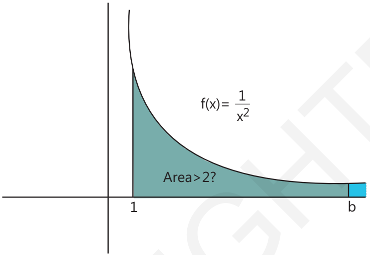
Areas of Unbounded Regions



Question: What is the area A of the unbounded region between $y = 0$, $x = 1$ and the graph of $f(x) = \frac{1}{x^2}$? Is it infinite? Is it greater than 2?

Recall: We know that

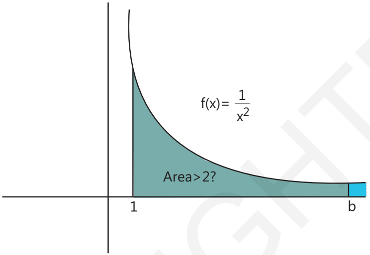
$$\int_1^b \frac{1}{x^2}\, dx$$

represents the area bounded by the lines $y = 0$, $x = 1$, $x = b$ and the graph of $f(x) = \frac{1}{x^2}$.

Question: Can we find a $b > 1$ so that

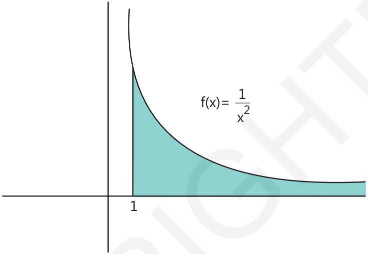
$$\int_1^b \frac{1}{x^2}\, dx > 2\ ?$$

Areas of Unbounded Regions



$$
\begin{aligned}
\int_1^b \frac{1}{x^2}\,dx &= \int_1^b x^{-2}\,dx \\
&= -\frac{1}{x}\Big|_1^b \\
&= -\frac{1}{b}+1 \\
&= 1-\frac{1}{b} \\
&< 2
\end{aligned}
$$

Areas of Unbounded Regions



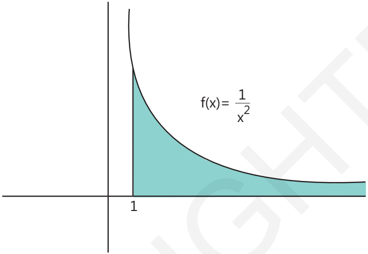
Strategy: Proceed as we did with infinite sums.
Let

$$A_b = \int_1^b \frac{1}{x^2}\, dx$$

and let $b \to \infty$ then define

$$A = \lim_{b\to\infty} A_b$$

Areas of Unbounded Regions



In particular, we would have

$$\begin{aligned} A &= \lim_{b\to\infty} \int_1^b \frac{1}{x^2}\,dx \\ &= \lim_{b\to\infty} 1 - \frac{1}{b} \\ &= 1 \end{aligned}$$

In this case, we would like to have

$$\int_1^\infty \frac{1}{x^2}\,dx = 1$$

Type I Improper Integrals

Definition: [Type I Improper Integrals]

1) Let f be integrable on $[a, b]$ for each $a \leq b$. We say that the *Type I improper integral*

$$\int_a^\infty f(x)\,dx$$

converges if

$$\lim_{b\to\infty} \int_a^b f(x)\,dx$$

exists.
In this case, we write

$$\int_a^\infty f(x)\,dx = \lim_{b\to\infty} \int_a^b f(x)\,dx$$

Otherwise, we say that $\int_a^\infty f(x)\,dx$ *diverges*.

Type I Improper Integrals

Definition: [Type I Improper Integrals]

2) Let f be integrable on $[b, a]$ for each $b \leq a$. We say that the *Type I improper integral*

$$\int_{-\infty}^{a} f(x)\, dx$$

converges if

$$\lim_{b \to -\infty} \int_{b}^{a} f(x)\, dx$$

exists.
In this case, we write

$$\int_{-\infty}^{a} f(x)\, dx = \lim_{b \to -\infty} \int_{b}^{a} f(x)\, dx$$

Otherwise, we say that $\int_{-\infty}^{a} f(x)\, dx$ diverges.

Type I Improper Integrals

Definition: [Type I Improper Integrals]

3) Assume that f is integrable on $[a, b]$ for each $a, b \in \mathbb{R}$ with $a < b$. We say that the *Type I improper integral*

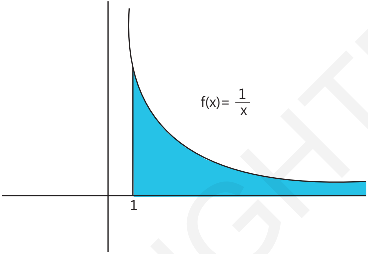
$$\int_{-\infty}^{\infty} f(x)\,dx$$

converges if both $\int_{-\infty}^{0} f(x)\,dx$ and $\int_{0}^{\infty} f(x)\,dx$ converge. In this case, we write

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{0} f(x)\,dx + \int_{0}^{\infty} f(x)\,dx$$

Otherwise, we say that $\int_{-\infty}^{\infty} f(x)\,dx$ diverges.

Type I Improper Integrals



Example: Show that $\int_1^\infty \frac{1}{x}\,dx$ diverges.

$$
\begin{aligned}
\int_1^\infty \frac{1}{x}\,dx &= \lim_{b\to\infty} \int_1^b \frac{1}{x}\,dx \\
&= \lim_{b\to\infty} \ln(x)\Big|_1^b \\
&= \lim_{b\to\infty} (\ln(b) - \ln(1)) \\
&= \lim_{b\to\infty} \ln(b) \\
&= \infty
\end{aligned}
$$

p-Test for Type I Improper Integrals

Question: For which values of p does the improper integral

$$\int_1^\infty \frac{1}{x^p}\,dx$$

converge?

Key Observation : If $\alpha > 0$, then

$$\lim_{b\to\infty} b^\alpha = \infty$$

and if $\alpha < 0$, then

$$\lim_{b\to\infty} b^\alpha = 0$$

p-Test for Type I Improper Integrals

Note: For any $b > 1$,

$$\begin{aligned}\int_1^b \frac{1}{x^p}\,dx &= \int_1^b x^{-p}\,dx \\ &= \left.\frac{x^{-p+1}}{-p+1}\right|_1^b \\ &= \frac{b^{1-p}}{1-p} - \frac{1}{-p+1} \\ &= \frac{b^{1-p}}{1-p} + \frac{1}{p-1}\end{aligned}$$

If $p < 1$, then $1 - p > 0$. Therefore,

$$\lim_{b\to\infty} \frac{b^{1-p}}{1-p} + \frac{1}{p-1} = \infty$$

and hence that $\int_1^\infty \frac{1}{x^p}\,dx$ diverges.

p-Test for Type I Improper Integrals

Note (continued): However, if $p > 1$, then $1 - p < 0$ and

$$\int_1^\infty \frac{1}{x^p}\,dx = \lim_{b\to\infty} \frac{b^{1-p}}{1-p} + \frac{1}{p-1} = \frac{1}{p-1}$$

Theorem: [p-Test for Type I Improper Integrals]

The improper integral

$$\int_1^\infty \frac{1}{x^p}\,dx$$

converges if and only if $p > 1$.

If $p > 1$, then

$$\int_1^\infty \frac{1}{x^p}\,dx = \frac{1}{p-1}$$

$\int_0^\infty e^{-x}\,dx$

Example: Evaluate $\int_0^\infty e^{-x}\,dx$.

We have

$$\begin{aligned}
\int_0^\infty e^{-x}\,dx &= \lim_{b\to\infty} \int_0^b e^{-x}\,dx \\
&= \lim_{b\to\infty} -e^{-x}\Big|_0^b \\
&= \lim_{b\to\infty} (-e^{-b} + e^0) \\
&= \lim_{b\to\infty} (-e^{-b} + 1) \\
&= 1
\end{aligned}$$

since $\lim_{b\to\infty} -e^{-b} = 0$.

Type I Improper Integrals

Theorem: [Type I Improper Integrals]

Assume that $\int_a^\infty f(x)\,dx$ and $\int_a^\infty g(x)\,dx$ both converge.

1. $\int_a^\infty cf(x)\,dx$ converges for each $c \in \mathbb{R}$ and
$$\int_a^\infty cf(x)\,dx = c\int_a^\infty f(x)\,dx$$
2. $\int_a^\infty (f(x)+g(x))\,dx$ converges and
$$\int_a^\infty (f(x)+g(x))\,dx = \int_a^\infty f(x)\,dx + \int_a^\infty g(x)\,dx$$
3. If $f(x) \leq g(x)$ for all $a \leq x$, then
$$\int_a^\infty f(x)\,dx \leq \int_a^\infty g(x)\,dx$$
4. If $\int_a^\infty f(x)\,dx$ converges and $a < c < \infty$, then $\int_c^\infty f(x)\,dx$ converges and
$$\int_a^\infty f(x)\,dx = \int_a^c f(x)\,dx + \int_c^\infty f(x)\,dx$$

Important Note

Important Note:

1) Always evaluate improper integrals **by first expressing them as limits of proper integrals**. That is

$$\int_a^\infty f(x)\,dx = \lim_{b\to\infty}\int_a^b f(x)\,dx$$

2) **Never** apply techniques of integration such as change of variables or integration by parts directly to an improper integral. Apply these techniques to the proper integral

$$\int_a^b f(x)\,dx$$

and then take the limit.