The Gamma Function

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The Gamma Function

Example: Evaluate

$$\int_0^\infty t^0 e^{-t} dt = \int_0^\infty e^{-t} dt$$

By definition

$$\begin{aligned}
\int_0^\infty e^{-t} dt &= \lim_{b\to\infty} \int_0^b e^{-t} dt \\
&= \lim_{b\to\infty} -e^{-t}\Big|_0^b \\
&= \lim_{b\to\infty} (-e^{-b}) - (-e^{-0}) \\
&= 1
\end{aligned}$$

The Gamma Function

Definition: [The Gamma Function]

For each $x \in \mathbb{R}$ define

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt$$

The function Γ is called the Gamma function.

The Gamma Function

Remarks:

1) We have just seen that $\Gamma(1) = 1$.

2) By modifying the previous example we can show that the improper integral

$$\int_0^\infty e^{-\frac{t}{2}}\, dt$$

also converges.

Next we observe that the Fundamental Log Limit shows that for any $x \in \mathbb{R}$ we have that

$$\lim_{t\to\infty} t^{x-1} e^{-\frac{t}{2}} = 0$$

so that if t is large enough

$$t^{x-1}e^{-t} = [t^{x-1}e^{-\frac{t}{2}}] \cdot e^{-\frac{t}{2}} < e^{-\frac{t}{2}}$$

By modifying the Comparison Test for Improper Integrals we can show that

$$\int_0^\infty t^{x-1}e^{-t}\, dt$$

is always convergent.

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Observation: If we apply the Integration by Parts formula we get the following interesting result:

$$\int t^x e^{-t} dt = -t^x e^{-t} + x \cdot \int t^{x-1} e^{-t} dt$$

It follows that

$$\begin{aligned}
\Gamma(x+1) &= \lim_{b\to\infty} \int_0^b t^x e^{-t} dt \\
&= \lim_{b\to\infty} -t^x e^{-t}|_0^b + x \cdot \int_0^b t^{x-1} e^{-t} dt \\
&= \lim_{b\to\infty} -b^x e^{-b} + x \cdot \lim_{b\to\infty} \int_0^b t^{x-1} e^{-t} dt \\
&= 0 + x \cdot \int_0^\infty t^{x-1} e^{-t} dt \\
&= x \cdot \Gamma(x)
\end{aligned}$$

Key Property: Since $\Gamma(1) = 1$, we have for each $n \in \mathbb{N}$

$$\Gamma(n) = (n-1)!$$

Remark:

The Γ function has many applications in physics, applied mathematics, number theory and other areas of pure mathematics, probability and statistics.