Method of Substitution: Examples

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Method of Substitution

Method of Substitution:

1) Start with

$$\int f(g(x))g'(x)\,dx.$$

2) Identify the possible substitution

$$u = g(x).$$

3) Differentiating both sides gives us

$$\frac{du}{dx} = g'(x).$$

4) If we treat du and dx as if they were "numbers", then

$$du = g'(x)\,dx.$$

We can now *substitute* u for $g(x)$ and du for $g'(x)dx$ to get

$$\int f(g(x))\,g'(x)\,dx = \int f(u)\,du \Big|_{u=g(x)}$$

which is the Change of Variables formula.

Method of Substitution

Example: Evaluate $\int x\cos(x^2)\,dx$.

Note: Using the substitution $u = x^2$ causes us to be out by a constant factor. However, this will not be a problem by proceeding as follows.

Let $u = x^2$ and $f(u) = \cos(u)$, then we get

$$du = 2x\,dx \qquad (*)$$

We can rearrange equation $(*)$ to get that

$$\frac{1}{2}\,du = x\,dx.$$

Substituting $\frac{1}{2}\,du$ for $x\,dx$ and u for x^2 we get

$$\begin{aligned}
\int x\cos(x^2)\,dx &= \int \frac{1}{2}\cos(u)\,du\,\Big|_{u=x^2} \\
&= \frac{1}{2}\int \cos(u)\,du\,\Big|_{u=x^2} \\
&= \frac{1}{2}\sin(u)\,\Big|_{u=x^2} + C \\
&= \frac{1}{2}\sin(x^2) + C
\end{aligned}$$

Method of Substitution

Example: Evaluate $\int \sec(\theta)\, d\theta$.

Remark: This does not appear to be an integral that can be evaluated by using substitution. However, we can make the following observation:

$$\begin{aligned} \sec(\theta) &= \sec(\theta)\left(\frac{\sec(\theta)+\tan(\theta)}{\sec(\theta)+\tan(\theta)}\right) \\ &= \frac{\sec^2(\theta)+\sec(\theta)\tan(\theta)}{\sec(\theta)+\tan(\theta)} \end{aligned}$$

so that

$$\int \sec(\theta)\, d\theta = \int \frac{\sec^2(\theta)+\sec(\theta)\tan(\theta)}{\sec(\theta)+\tan(\theta)}\, d\theta.$$

Method of Substitution

Example (continued): To evaluate

$$\int \sec(\theta)\,d\theta = \int \frac{\sec^2(\theta) + \sec(\theta)\tan(\theta)}{\sec(\theta) + \tan(\theta)}\,d\theta$$

let $u = \sec(\theta) + \tan(\theta)$ so

$$du = \sec^2(\theta) + \sec(\theta)\tan(\theta)d\theta.$$

We get

$$\begin{aligned}
\int \sec(\theta)\,d\theta &= \int \frac{\sec^2(\theta) + \sec(\theta)\tan(\theta)}{\sec(\theta) + \tan(\theta)}\,d\theta \\
&= \int \frac{du}{u} \\
&= \ln(|u|) + C \\
&= \ln(|\sec(\theta) + \tan(\theta)|) + C
\end{aligned}$$