Properties of the Integral

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Definition of the Integral

Definition: [Definite Integral]

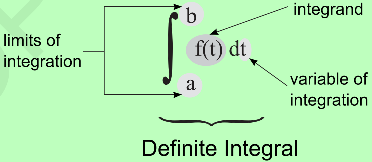
We say that a bounded function f is *integrable* on $[a, b]$ if there exists a unique number $I \in \mathbb{R}$ such that if whenever $\{P_n\}$ is a sequence of partitions with $\lim_{n\to\infty} \|P_n\| = 0$ and $\{S_n\}$ is any sequence of Riemann sums associated with the P_n's, we have

$$\lim_{n\to\infty} S_n = I.$$

In this case, we call I the *integral* of f over $[a, b]$ and denote it by

$$\int_a^b f(t)\ dt$$

The points a and b are called the *limits of integration* and the function $f(t)$ is called the *integrand*. The variable t is called the *variable of integration*.



Properties of Definite Integrals

Theorem: [Properties of Definite Integrals]

Assume that f and g are integrable on the interval $[a, b]$. Then

i) For any $c \in \mathbb{R}$, $\int_a^b c\, f(t)\ dt = c \int_a^b f(t)\ dt$

ii) $\int_a^b (f+g)(t)\ dt = \int_a^b f(t)\ dt + \int_a^b g(t)\ dt$

iii) If $m \leq f(t) \leq M$ for all $t \in [a, b]$, then

$$m(b-a) \leq \int_a^b f(t)\ dt \leq M(b-a)$$

iv) If $0 \leq f(t)$ for all $t \in [a, b]$, then $0 \leq \int_a^b f(t)\ dt$

v) If $g(t) \leq f(t)$ for all $t \in [a, b]$, then $\int_a^b g(t)\ dt \leq \int_a^b f(t)\ dt$.

vi) The function $|f|$ is integrable on $[a, b]$ and

$$\left| \int_a^b f(t)\, dt \right| \leq \int_a^b |\ f(t)\ |\ dt$$

Properties of Definite Integrals

Remark: So far in defining the definite integral we have always considered integrals of the form

$$\int_a^b f(t)\,dt$$

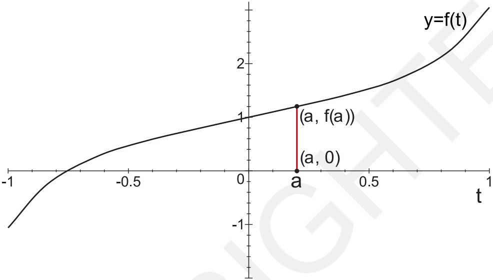
where $a < b$. However, it is necessary to give meaning to

$$\int_a^a f(t)\,dt$$

and to

$$\int_b^a f(t)\,dt.$$

Properties of Definite Integrals



Question: How should we define $\int_a^a f(t)\,dt$?

Remark: We can see that the line segment has height $f(a)$ but length 0. As such it makes sense to define its "area" to be 0.

Definition: [Identical Limits of Integration: $\int_a^a f(t)\,dt$]

Let $f(t)$ be defined at $t = a$. Then we define

$$\int_a^a f(t)\,dt = 0.$$

Properties of Definite Integrals

Remark: In the definition of

$$\int_a^b f(t)\,dt$$

where $a < b$, we began at the left-hand endpoint a of an interval $[a, b]$ and moved to the right towards b. In the case of the integral

$$\int_b^a f(t)\,dt$$

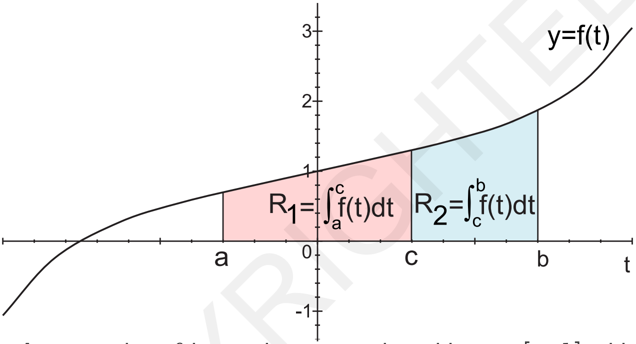
where $a < b$, we are suggesting that using the interval $[a, b]$ we move from b to the left towards a. This is the opposite or *negative* of the original orientation.

Definition: [Switching the Limits of Integration]

Let f be integrable on the interval $[a, b]$ where $a < b$. Then we define

$$\int_b^a f(t)\,dt = -\int_a^b f(t)\,dt.$$

Properties of Definite Integrals



Remark: Assume that f is continuous and positive on $[a, b]$ with $a < c < b$. Since

$$R = R_1 + R_2$$

we should have

$$\int_a^b f(t)\, dt = \int_a^c f(t)\, dt + \int_c^b f(t)\, dt$$

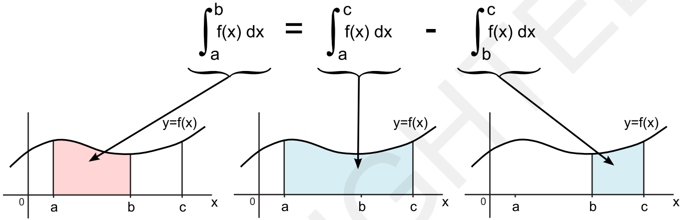
Properties of Definite Integrals

Theorem: [Integrals over Subintervals]

Assume that f is integrable on an interval I containing a, b and c. Then

$$\int_a^b f(t)\,dt = \int_a^c f(t)\,dt + \int_c^b f(t)\,dt.$$

Properties of Definite Integrals



Remark: Assume that f is integrable on the interval $[a, c]$ where $a < b < c$. Since $\int_b^c f(x)\,dx = -\int_c^b f(x)\,dx$, we get

$$\begin{aligned}\int_a^b f(x)\,dx &= \int_a^c f(x)\,dx - \int_b^c f(x)\,dx \\ &= \int_a^c f(x)\,dx + \int_c^b f(x)\,dx\end{aligned}$$