Introduction to Riemann Sums

Created by

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Area Under Curves

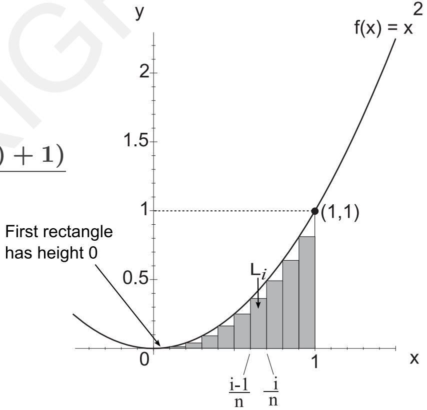
Let

$$0 = x_0 < x_1 < x_2 < x_3 < \cdots < x_{i-1} < x_i < \cdots < x_n = 1$$

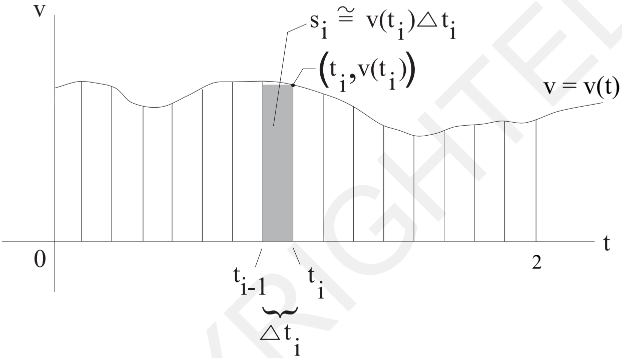
so that $x_i = \frac{i}{n}$. Then

$$\sum_{i=1}^{n} L_i = \sum_{i=1}^{n} (\frac{i-1}{n})^2 \frac{1}{n}$$

$$= \frac{1}{n^3} \frac{(n-1)(n+1-1)(2(n-1)+1)}{6}$$



Displacement versus Velocity



Divide the 2 hour duration of the trip into 120 one minute intervals

$$0 = t_0 < t_1 < t_2 < t_3 < \cdots < t_{i-1} < t_i < \cdots < t_{120} = 2$$

so that $t_i = \frac{2i}{120} = \frac{i}{60}$ hours.

$$s = \sum_{i=1}^{120} s_i \cong \sum_{i=1}^{120} v(t_i)\Delta t_i = \sum_{i=1}^{120} v(t_i)\frac{2}{120}$$

Introduction to Riemann Sums

Definition: [Partition]

A *partition* P for the interval $[a, b]$ is a finite increasing sequence of numbers of the form

$$a = t_0 < t_1 < t_2 < \cdots < t_{i-1} < t_i < \cdots < t_{n-1} < t_n = b.$$

Note: A partition subdivides the interval $[a, b]$ into n subintervals

$$[t_0, t_1], [t_1, t_2], \cdots, [t_{i-1}, t_i], \cdots, [t_{n-2}, t_{n-1}], [t_{n-1}, t_n].$$

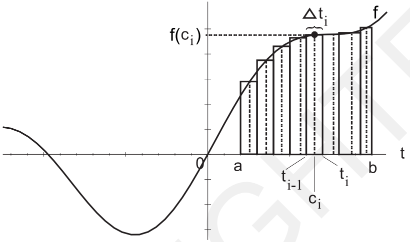
Let

$$\Delta t_i = t_i - t_{i-1}$$

and let the *norm* of the partition P be

$$\|P\| = \max\{\Delta t_1, \Delta t_2, \ldots, \Delta t_n\}.$$

Introduction to Riemann Sums



Definition: [Riemann Sum]

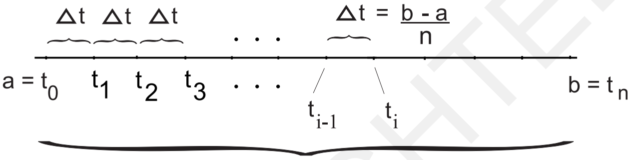
Given a bounded function f on $[a, b]$, a partition P

$$a = t_0 < t_1 < t_2 < \cdots < t_{i-1} < t_i < \cdots < t_{n-1} < t_n = b$$

of $[a, b]$, and a set $\{c_1, c_2, \ldots, c_n\}$ where $c_i \in [t_{i-1}, t_i]$, then a *Riemann sum* for f with respect to P is a sum of the form

$$S = \sum_{i=1}^{n} f(c_i)\Delta t_i.$$

Regular n-Partition



Definition: [Regular n-Partition]

Given an interval $[a, b]$ and an $n \in \mathbb{N}$, the *regular* $n-$*partition* of $[a, b]$ is the partition $P^{(n)}$ with

$$a = t_0 < t_1 < t_2 < \cdots < t_{i-1} < t_i < \cdots < t_{n-1} < t_n = b$$

of $[a, b]$ where each subinterval has the *same* length $\Delta t_i = \frac{b-a}{n}$. Hence,

$$t_i = a + i \cdot \frac{b-a}{n}$$

Right-hand Riemann Sum

Definition: [Right-hand Riemann Sum]

The *right-hand Riemann sum* for f with respect to the partition P is the Riemann sum R obtained from P by choosing c_i to be t_i, the right-hand endpoint of $[t_{i-1}, t_i]$. That is

$$R = \sum_{i=1}^{n} f(t_i)\Delta t_i.$$

If $P^{(n)}$ is the regular n-partition, we denote the right-hand Riemann sum by

$$\begin{aligned} R_n = \sum_{i=1}^{n} f(t_i)\Delta t_i &= \sum_{i=1}^{n} f(t_i)\frac{b-a}{n} \\ &= \sum_{i=1}^{n} f\left(a + i\left(\frac{b-a}{n}\right)\right)\left(\frac{b-a}{n}\right) \end{aligned}$$

Left-hand Riemann Sum

Definition: [Left-hand Riemann Sum]

The *left-hand Riemann sum* for f with respect to the partition P is the Riemann sum L obtained from P by choosing c_i to be t_{i-1}, the left-hand endpoint of $[t_{i-1}, t_i]$. That is

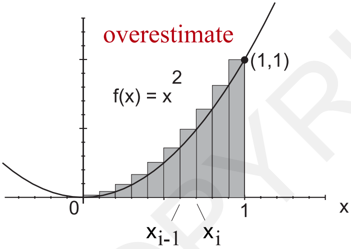
$$L = \sum_{i=1}^{n} f(t_{i-1})\Delta t_i.$$

If $P^{(n)}$ is the regular n-partition, we denote the left-hand Riemann sum by

$$\begin{aligned} L_n = \sum_{i=1}^{n} f(t_{i-1})\Delta t_i &= \sum_{i=1}^{n} f(t_{i-1})\frac{b-a}{n} \\ &= \sum_{i=1}^{n} f\left(a + (i-1)\left(\frac{b-a}{n}\right)\right)\left(\frac{b-a}{n}\right) \end{aligned}$$

Right-hand versus Left-hand Riemann Sum

(a) Right-hand Riemann sum using right endpoints



(b) Left-hand Riemann sum using left endpoints