Fundamental Theorem of Calculus (Part 2)

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Example: Evaluate

\[ \int_{0}^{2} t^3 \, dt. \]

Observation: If

\[ G(x) = \int_{0}^{x} t^3 \, dt, \]

then

\[ G(2) = \int_{0}^{2} t^3 \, dt. \]

We know from the FTC1 that

\[ G'(x) = x^3 \]

so \( G \) is an antiderivative of \( x^3 \). Hence there exists a constant \( C \) such that

\[ G(x) = \int_{0}^{x} t^3 \, dt = \frac{x^4}{4} + C. \]

Question: How does this help us?
Fundamental Theorem of Calculus (Part 2)

**Note:** We have just seen that

\[ G(x) = \int_0^x t^3 \, dt = \frac{x^4}{4} + C \]

for some constant \( C \in \mathbb{R} \).

However, we also know that

\[ 0 = \int_0^0 t^3 \, dt = G(0) = \frac{0^4}{4} + C = C \]

so

\[ G(x) = \int_0^x t^3 \, dt = \frac{x^4}{4}. \]

Finally,

\[ \int_0^2 t^3 \, dt = G(2) = \frac{2^4}{4} = 4. \]

**Question:** Did we really need to find \( C \)?
Key Observation:

Let $F$ and $G$ be any two antiderivatives of the same function $f$. Then

$$G(x) = F(x) + C.$$ 

Let $a, b \in \mathbb{R}$. Then

$$G(b) - G(a) = (F(b) + C) - (F(a) + C)$$

$$= F(b) - F(a).$$
Fundamental Theorem of Calculus (Part 2)

Key Observation (continued):
Assume that \( f \) is continuous and that we want to know
\[
\int_{a}^{b} f(t) \, dt.
\]

Let
\[
G(x) = \int_{a}^{x} f(t) \, dt.
\]

Then \( G \) is an antiderivative of \( f \) and if \( F \) is any other antiderivative of \( f \), then
\[
\int_{a}^{b} f(t) \, dt = G(b) = G(b) - G(a) \quad \text{(since } G(a) = 0 \text{)}
\]
\[
= F(b) - F(a)
\]
Fundamental Theorem of Calculus (Part 2)

Example: Evaluate

\[ \int_0^2 t^3 \, dt. \]

Solution: We know that

\[ F(x) = \frac{x^4}{4} \]

is an antiderivative for \( f(x) = x^3 \).

Hence

\[ \int_0^2 t^3 \, dt = F(2) - F(0) \]

\[ = \frac{2^4}{4} - \frac{0^4}{4} \]

\[ = \frac{16}{4} - \frac{0}{4} \]

\[ = 4 \]
Theorem: [Fundamental Theorem of Calculus (Part 2) [FTC2]]
Assume that $f$ is continuous and that $F$ is any antiderivative of $f$. Then
\[
\int_{a}^{b} f(t) dt = F(b) - F(a).
\]

Notation: We write
\[
F(x) \bigg|_{a}^{b} = F(b) - F(a)
\]
Example: Evaluate

\[ \int_{0}^{\pi} \sin(t) \, dt. \]

Solution: Since \( f(t) = \sin(t) \) is continuous and \( F(t) = -\cos(t) \) is an antiderivative of \( f \), by the FTC2, we have

\[
\int_{0}^{\pi} \sin(t) \, dt = -\cos(t) \bigg|_{0}^{\pi}
\]

\[ = -\cos(\pi) - (-\cos(0)) \]

\[ = 2 \]
Example: Evaluate

$$\int_{-\pi}^{\pi} \sin(t) \, dt$$

Solution: Using a geometrical argument, this integral should equal 0.

$$\int_{-\pi}^{\pi} \sin(t) \, dt = -\cos(t) \bigg|_{-\pi}^{\pi}$$

$$= -\cos(\pi) - (-\cos(-\pi))$$

$$= 0$$
Fundamental Theorem of Calculus (Part 2)

Key Remark: It is important that we emphasize the difference between the meaning of

\[ \int_{a}^{b} f(t) \, dt \quad \text{and} \quad \int f(t) \, dt \]

The first expression, \( \int_{a}^{b} f(t) \, dt \)

is called a *definite* integral. It represents a *number* that is defined as a limit of Riemann sums.

The second expression, \( \int f(t) \, dt \)

is called an *indefinite* integral. It represents the *family of all functions that are antiderivatives* of the given function \( f \).