Displacement versus Velocity

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Recall:
We know that if \( s(t) \) represents the displacement of an object and \( v(t) \) represents its velocity, then

\[
\frac{ds}{dt} = s'(t) = v(t)
\]

Problem:
Suppose that a car travels for two hours along a highway. The odometer is broken but the speedometer is working. How can we determine the distance traveled from only the data about the car’s velocity?
Step 1: Suppose that we always travel forward on the highway at a constant velocity of 90 km/hr. If $s = \text{displacement}$, $v = \text{velocity}$, and $\Delta t = \text{time elapsed}$, then

$$s = v \Delta t$$

Key Observation: In this case, displacement equals the area under the graph of $v$ from $t = 0$ to $t = 2$. 
Step 2: Divide the 2 hour duration of the trip into 120 one minute intervals

\[0 = t_0 < t_1 < t_2 < t_3 < \cdots < t_{i-1} < t_i < \cdots < t_{120} = 2 \text{ hours}\]

so that \( t_i = \frac{2i}{120} = \frac{i}{60} \) hours.

Let \( s_i \) be the displacement (distance traveled) during time \( t_{i-1} \) until \( t_i \), the \( i \)-th minute of the trip.

If \( s \) is the total displacement, we have

\[ s = s_1 + s_2 + s_3 + \cdots + s_i + \cdots + s_{120} \]

\[ = \sum_{i=1}^{120} s_i \]
Step 2 (continued): If we assume that the velocity does not vary much over any one minute period, we can assume that the velocity during the interval \([t_{i-1}, t_i]\) was the same as it was at \(t_i\) so that

\[ s_i \approx v(t_i) \Delta t_i = v(t_i) \frac{2}{120} \]

Then we have the estimate for \(s\):

\[ s = \sum_{i=1}^{120} s_i \approx \sum_{i=1}^{120} v(t_i) \Delta t_i = \sum_{i=1}^{120} v(t_i) \frac{2}{120} \]
Step 3: Divide the two hour period into 7200 subintervals each of length 1 second to get

\[ s_i \approx v(t_i) \Delta t_i = v(t_i) \frac{2}{7200} \]

and

\[ s = \sum_{i=1}^{7200} s_i \approx \sum_{i=1}^{7200} v(t_i) \Delta t_i = \sum_{i=1}^{7200} v(t_i) \frac{2}{7200} \]
**Step 4:** Divide the two hour period into \( n \) subintervals each of length \( \frac{2}{n} \) hours with \( t_i = \frac{2i}{n} \) to get

\[
s_i \approx v(t_i) \Delta t_i = v(t_i) \frac{2}{n}
\]

and

\[
s = \sum_{i=1}^{n} s_i \approx S_n = \sum_{i=1}^{n} v(t_i) \Delta t_i = \sum_{i=1}^{n} v(t_i) \frac{2}{n}.
\]
Step 5: With

\[ S_n = \sum_{i=1}^{n} v(t_i) \Delta t_i = \sum_{i=1}^{n} v(t_i) \frac{2}{n} \]

it can be shown that the sequence \( \{S_n\} \) converges and that

\[ \lim_{n \to \infty} \{S_n\} = s. \]
Geometric Interpretation: Displacement versus Velocity
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\[ s_i = v(t_i) \Delta t_i \]

= area of rectangle

\[ v = v(t) \]

\[ 0 \quad t_{i-1} \quad t_i \quad 2 \]

\[ \Delta t_i \]
Geometric Interpretation: Displacement versus Velocity

\[ S_n = \text{sum of areas of all rectangles} \]
Geometric Interpretation:
Displacement versus Velocity

Distance travelled (S) = Area under curve

Conclusion: Displacement is the area under the velocity graph!