Definition of the Integral

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Definition of the Integral

\[ R_n = \frac{1}{n^3} \sum_{i=1}^{n} i^2 \]

\[ = \frac{1}{n^3} \frac{(n)(n + 1)(2(n) + 1)}{6} \]

\[ = \frac{2 + \frac{3}{n} + \frac{1}{n^2}}{6} \to \frac{1}{3} \]

\[ f(x) = x^2 \]

\(R \rightarrow (1, 1)\)
Definition of the Integral

\[ L_n = \sum_{i=1}^{n} \left( \frac{i - 1}{n} \right)^2 \cdot \frac{1}{n} \]

\[ = \frac{2 - \frac{3}{n} + \frac{1}{n^2}}{6} \to \frac{1}{3} \]

If \( \{S_n\} \) is any sequence of Riemann sums for \( P(n) \), then

\[ L_n \leq S_n \leq R_n \Rightarrow \lim_{n \to \infty} S_n = \frac{1}{3} \]
Definition of the Integral

Definition: [Definite Integral]

We say that a bounded function $f$ is integrable on $[a, b]$ if there exists a unique number $I \in \mathbb{R}$ such that if whenever $\{P_n\}$ is a sequence of partitions with $\lim_{n \to \infty} \|P_n\| = 0$ and $\{S_n\}$ is any sequence of Riemann sums associated with the $P_n$’s, we have

$$\lim_{n \to \infty} S_n = I.$$

In this case, we call $I$ the integral of $f$ over $[a, b]$ and denote it by

$$\int_a^b f(t) \, dt$$

The points $a$ and $b$ are called the limits of integration and the function $f(t)$ is called the integrand. The variable $t$ is called the variable of integration.
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**Note:** The variable of integration is sometimes called a *dummy variable* in the sense that if we were to replace $t$’s by $x$’s everywhere, we would not change the value of the integral.

**Question:** Are all bounded functions on $[a, b]$ integrable?

**Answer:** No. Let

$$f(x) = \begin{cases} 
1 & \text{if } x \in \mathbb{Q}, \\
-1 & \text{if } x \notin \mathbb{Q}.
\end{cases}$$

Then $f$ is not integrable on any interval $[a, b]$. 
The Integrability Theorem

**Theorem: [The Integrability Theorem for Continuous Functions]**

Let $f$ be continuous on $[a, b]$. Then $f$ is integrable on $[a, b]$. Moreover,

$$
\int_{a}^{b} f(t) \, dt = \lim_{n \to \infty} S_n
$$

where

$$
S_n = \sum_{i=1}^{n} f(c_i) \Delta t_i
$$

is any Riemann sum associated with the regular $n$-partitions. In particular,

$$
\int_{a}^{b} f(t) \, dt = \lim_{n \to \infty} R_n = \lim_{n \to \infty} \sum_{i=1}^{n} f(t_i) \frac{b-a}{n}
$$

and

$$
\int_{a}^{b} f(t) \, dt = \lim_{n \to \infty} L_n = \lim_{n \to \infty} \sum_{i=1}^{n} f(t_{i-1}) \frac{b-a}{n}.
$$
The Integrability Theorem

Example: Since

\[ R_n = \frac{1}{n^3} \sum_{i=1}^{n} i^2 \]

\[ = \frac{2 + \frac{3}{n} + \frac{1}{n^2}}{6} \]

we get

\[ \int_{0}^{1} x^2 \, dx = \lim_{n \to \infty} \frac{2 + \frac{3}{n} + \frac{1}{n^2}}{6} = \frac{1}{3} \]
Integrating Constant Functions

Example:
Let \( f(t) = \alpha \) for each \( t \in [a, b] \). Then

\[
R_n = \sum_{i=1}^{n} f(t_i) \Delta t_i \\
= \sum_{i=1}^{n} \alpha \left( \frac{b - a}{n} \right) \\
= \alpha \sum_{i=1}^{n} \left\{ \frac{b - a}{n} \right\} \\
= \alpha \left( b - a \right)
\]

Since \( R_n = \alpha (b - a) \) for each \( n \),

\[
\int_{a}^{b} \alpha \, dt = \alpha (b - a).
\]
Properties of Definite Integrals

Theorem: [Properties of Definite Integrals]
Assume that \( f \) and \( g \) are integrable on the interval \([a, b]\). Then

i) For any \( c \in \mathbb{R} \), \( \int_a^b c \, f(t) \, dt = c \int_a^b f(t) \, dt \)

ii) \( \int_a^b (f + g)(t) \, dt = \int_a^b f(t) \, dt + \int_a^b g(t) \, dt \)

iii) If \( m \leq f(t) \leq M \) for all \( t \in [a, b] \), then

\[
m(b - a) \leq \int_a^b f(t) \, dt \leq M(b - a)
\]

iv) If \( 0 \leq f(t) \) for all \( t \in [a, b] \), then \( 0 \leq \int_a^b f(t) \, dt \).

v) If \( g(t) \leq f(t) \) for all \( t \in [a, b] \), then \( \int_a^b g(t) \, dt \leq \int_a^b f(t) \, dt \).

vi) The function \( |f| \) is integrable on \([a, b]\) and

\[
\left| \int_a^b f(t) \, dt \right| \leq \int_a^b |f(t)| \, dt
\]
Properties of Definite Integrals

Proof of (iii): Assume that

\[ m \leq f(t) \leq M \]

for all \( t \in [a, b] \). Let

\[ a = t_0 < t_1 < t_2 < \cdots < t_{i-1} < t_i < \cdots < t_{n-1} < t_n = b \]

be any partition of \([a, b]\). Then

\[ m(b - a) = \sum_{i=1}^{n} m \Delta t_i \leq \sum_{i=1}^{n} f(t_i) \Delta t_i \leq \sum_{i=1}^{n} M \Delta t_i = M(b - a) \]

since

\[ \sum_{i=1}^{n} \Delta t_i = b - a. \]