Change of Variables for the Definite Integral

Created by

Barbara Forrest and Brian Forrest

Change of Variables for the Indefinite Integral

Recall:

Definition: [Change of Variables Formula]

Let $u = g(x)$ be differentiable. Then

$$\int f(g(x))g'(x)\,dx = \int f(u)\,du.$$

Question: How does the Change of Variables formula work with

$$\int_a^b f(g(x))g'(x)\,dx?$$

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Key Observation: Suppose that we want to evaluate

$$\int_a^b f(g(x))g'(x)\,dx$$

where f and g' are continuous functions.

We know that if $h(u)$ is an antiderivative of $f(u)$, then

$$H(x) = h(g(x))$$

is an antiderivative of

$$f(g(x))g'(x).$$

This means that we can apply the Fundamental Theorem of Calculus Part 2 to get

$$\begin{aligned}\int_a^b f(g(x))g'(x)\,dx &= H(b) - H(a)\\ &= h(g(b)) - h(g(a))\\ &= \int_{g(a)}^{g(b)} f(u)\,du\end{aligned}$$

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Theorem: [Change of Variables]

Assume that $g'(x)$ is continuous on $[a, b]$ and $f(u)$ is continuous on $g([a, b])$, then

$$\int_a^b f(g(x))g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du.$$

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Theorem: [Change of Variables]

Assume that $g'(x)$ is continuous on $[a, b]$ and $f(u)$ is continuous on $g([a, b])$, then

$$\int_a^b f(g(x))g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du.$$

Note: We will often write

$$\int_{x=a}^{x=b} f(g(x))g'(x)\,dx = \int_{u=g(a)}^{u=g(b)} f(u)\,du.$$

to emphasize which limits of integration correspond to each variable.

Change of Variables for the Definite Integral

Example: Evaluate $\int_2^4 (5x-6)^3\,dx$.

Solution: Let $u = g(x) = 5x - 6$. Then

$$du = g'(x)\,dx = 5\,dx,$$

and we have

$$\frac{1}{5}\,du = dx.$$

The Change of Variables Theorem shows us that

$$\int_2^4 (5x-6)^3\,dx \quad = \quad \int_{u=g(2)}^{u=g(4)} u^3\,\frac{1}{5}du$$

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Example (continued):

Now since

$$g(a) = g(2) = 5(2) - 6 = 4 \quad \text{and} \quad g(b) = g(4) = 5(4) - 6 = 14$$

we have

$$\begin{aligned} \int_2^4 (5x-6)^3\,dx &= \frac{1}{5}\int_4^{14} u^3\,du \\ &= \frac{1}{5}\left(\frac{1}{4}u^4\right)\Bigg|_4^{14} \\ &= \frac{1}{20}(14^4 - 4^4) \\ &= 1908 \end{aligned}$$

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Example: Evaluate $\displaystyle\int_0^1 \frac{x\,dx}{\sqrt{x^2+1}}$.

Let $u = g(x) = x^2 + 1$. Then

$$du = g'(x)\,dx = 2x\,dx.$$

The Change of Variables Theorem shows us that

$$\begin{aligned}
\int_0^1 \frac{x\,dx}{\sqrt{x^2+1}} &= \int_{u=g(0)}^{u=g(1)} (u^{-\frac{1}{2}})\,(\tfrac{1}{2}du) \\
&= \frac{1}{2}\int_1^2 u^{-\frac{1}{2}}\,du \\
&= \frac{1}{2}\left(2\,u^{\frac{1}{2}}\right)\Big|_1^2 \\
&= 2^{\frac{1}{2}} - 1^{\frac{1}{2}} \\
&= \sqrt{2} - 1
\end{aligned}$$

Change of Variables for the Definite Integral

Example: Evaluate $\int_0^{\frac{\pi}{4}} -2\cos^2(2x)\ \sin(2x)\, dx$.

Let $u = g(x) = \cos(2x)$ and $f(u) = u^2$. Then since

$$du = g'(x)\, dx = -2\sin(2x)\, dx$$

the Change of Variables Theorem shows us that

$$\begin{aligned}
\int_0^{\frac{\pi}{4}} -2\cos^2(2x)\sin(2x)\, dx &= \int_{\cos(2(0))}^{\cos(2(\frac{\pi}{4}))} u^2\, du \\
&= \int_{\cos(0)}^{\cos(\frac{\pi}{2})} u^2\, du \\
&= \int_1^0 u^2\, du \\
&= \left.\frac{u^3}{3}\right|_1^0 \\
&= \frac{0^3}{3} - \frac{1^3}{3} = -\frac{1}{3}
\end{aligned}$$