Taylor’s Theorem

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Taylor Polynomials

Definition: [Taylor Polynomials]

Assume that $f(x)$ is $n$-times differentiable at $x = a$. The $n$-th degree Taylor polynomial for $f(x)$ centered at $x = a$ is the polynomial

$$T_{n,a}(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x - a)^k$$

$$= f(a) + f'(a)(x - a) + \frac{f''(a)}{2!}(x - a)^2 + $$

$$\cdots + \frac{f^{(n)}(a)}{n!}(x - a)^n$$

Observation: We have seen that for functions $f(x)$ such as $\cos(x)$, $\sin(x)$ and $e^x$, that

$$T_{n,a}(x) \approx f(x)$$

near $x = a$. 
Taylor’s Remainder

**Definition: [Taylor Remainder]**

Assume that $f(x)$ is $n$ times differentiable at $x = a$. Let

$$R_{n,a}(x) = f(x) - T_{n,a}(x).$$

$R_{n,a}(x)$ is called the $n$-th degree Taylor remainder function centered at $x = a$.

**Note:** The error in using a Taylor Polynomial to approximate $f(x)$ is given by

$$\text{Error} = | R_{n,a}(x) |.$$

**Central Problem:** Given a function $f(x)$ and a point $x = a$, how do we estimate the size of the Taylor Remainder $R_{n,a}(x)$?
Taylor’s Theorem

Theorem: [Taylor’s Theorem]

Assume that \( f(x) \) is \( n + 1 \)-times differentiable on an interval \( I \) containing \( x = a \). Let \( x \in I \). Then there exists a point \( c \) between \( x \) and \( a \) such that

\[
f(x) - T_{n,a}(x) = R_{n,a}(x) = \frac{f^{(n+1)}(c)}{(n+1)!} (x - a)^{n+1}.
\]

Remarks:

1) Since \( T_{0,a}(x) = f(a) \), when \( n = 0 \)

\[
f(x) - T_{0,a}(x) = f(x) - f(a) = f'(c)(x - a)
\]

which is the Mean Value Theorem.

2) Since \( T_{1,a}(x) = L^f_a(x) \),

\[
|f(x) - L^f_a(x)| = |R_{1,a}(x)| = \frac{|f''(c)|}{2} (x - a)^2
\]

is the error in using the linear approximation.
Remarks (continued):

3) Taylor’s Theorem does not tell us how to find the point $c$, only that it exists. Therefore, to estimate $R_{n,a}(x)$ we must first estimate how large $|f^{(n+1)}(c)|$ could be without knowing the value of $c$. 
Taylor’s Theorem

**Example:** Use linear approximation to estimate \( \sin(0.01) \) and show that the error in using this approximation is less than \( 10^{-4} \).

**Solution:** We know that \( f(0) = \sin(0) = 0 \) and that \( f'(0) = \cos(0) = 1 \), so

\[
L_0(x) = T_{1,0}(x) = x.
\]

Therefore,

\[
\sin(0.01) \cong L_0(0.01) = 0.01
\]

Since \( f(x) = \sin(x) \), \( f'(x) = \cos(x) \), and \( f''(x) = -\sin(x) \), Taylor’s Theorem guarantees that there exists some \( c \) between 0 and 0.01 such that the error in the linear approximation is given by

\[
|R_{1,0}(0.01)| = \left| \frac{f''(c)}{2} (0.01 - 0)^2 \right| = \left| \frac{-\sin(c)}{2} (0.01)^2 \right| \leq \frac{(0.01)^2}{2}
\]

since \( | - \sin(c) | \leq 1 \).
Taylor’s Theorem

Remark: For \( f(x) = \sin(x) \), we saw that

\[ L_0(x) = T_{1,0}(x) = x. \]

However, since \( f''(0) = -\sin(0) = 0 \), we also have

\[ L_0(x) = T_{1,0}(x) = T_{2,0}(x) \]

so there exists a \( c \) between 0 and .01 such that

\[
| \sin(.01) - .01 | = | R_{2,0}(.01) |
\]

\[
= \left| \frac{f'''(c)}{6} (.01 - 0)^3 \right|
\]

\[
= \left| \frac{-\cos(c)}{6} (.01)^3 \right|
\]

\[
< 10^{-6}
\]

since \( | - \cos(c) | \leq 1 \).
Taylor’s Theorem

The statement is clearly true for $x = 0$. Let $x \in (0, \frac{\pi}{2}]$. Then by Taylor’s Theorem there is a $c \in (0, x)$ with

$$\sin(x) - x = R_{1,0}(x) = \frac{-\sin(c)}{2}x^2 < 0$$

since $\sin(c) > 0$ for any $c \in (0, \frac{\pi}{2})$. 

**Remark:** It can be shown that

$$\sin(x) \leq x$$

for all $x \geq 0$.

We can use Taylor’s Theorem to show this for all $x \in [0, \frac{\pi}{2}]$. 

The graph shows the function $f(x) = \sin(x)$ and the line $y = x$. The line $y = x$ intersects the graph of $f(x)$ at $x = 0$. For $x > 0$, the value of $\sin(x)$ is less than $x$.