Arithmetic with Big-O Notation
Part 2

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Recall: If,

\[ f(x) = O(x^n) \quad \text{and} \quad g(x) = O(x^m) \quad \text{as} \quad x \to 0, \]

then

\[ f(x) + g(x) = O(x^k) \quad \text{as} \quad x \to 0 \]

where

\[ k = \min\{n, m\}. \]

We write

\[ O(x^n) + O(x^m) = O(x^k) \]

where

\[ k = \min\{n, m\}. \]
Theorem: [Arithmetic of Big-O]

Assume that \( f(x) = O(x^n) \) and \( g(x) = O(x^m) \) as \( x \to 0 \), for some \( m, n \in \mathbb{N} \). Let \( k \in \mathbb{N} \). Then we have the following:

1) \( c(O(x^n)) = O(x^n) \). That is, \( (cf)(x) = c \cdot f(x) = O(x^n) \).

2) \( O(x^n) + O(x^m) = O(x^k) \), where \( k = \min\{n, m\} \). That is, \( f(x) \pm g(x) = O(x^k) \).

3) \( O(x^n)O(x^m) = O(x^{n+m}) \). That is, \( f(x)g(x) = O(x^{n+m}) \).

4) If \( k \leq n \), then \( f(x) = O(x^k) \).

5) If \( k \leq n \), then \( \frac{1}{x^k} O(x^n) = O(x^{n-k}) \). That is, \( \frac{f(x)}{x^k} = O(x^{n-k}) \).

6) \( f(u^k) = O(u^{kn}) \). That is, we can simply substitute \( x = u^k \).

Note: In fact (5) is true if we replace \( x^k \) by \( x^\alpha \) for any \( \alpha \in \mathbb{R} \).
Big-O Arithmetic

Example: Show that \( f(x) = \cos(x^2) - 1 = -\frac{x^4}{2} + \mathcal{O}(x^8) \). Use this result to evaluate

\[
\lim_{x \to 0} \frac{\cos(x^2) - 1}{x^4}.
\]

Solution: We begin by observing that if \( g(u) = \cos(u) \), then since the third degree Taylor polynomial for \( g(u) \) centered at \( u = 0 \) is

\[
T_{3,0}(u) = 1 - \frac{u^2}{2}.
\]

Taylor’s Approximation Theorem II gives us that

\[
g(u) = 1 - \frac{u^2}{2} + \mathcal{O}(u^4).
\]

Arithmetic Rule (6) allows us to substitute \( x^2 \) for \( u \) to get

\[
\cos(x^2) = g(x^2) = 1 - \frac{(x^2)^2}{2} + \mathcal{O}((x^2)^4) = 1 - \frac{x^4}{2} + \mathcal{O}(x^8).
\]

Then

\[
f(x) = \cos(x^2) - 1 = -\frac{x^4}{2} + \mathcal{O}(x^8).
\]
Big-O Arithmetic

Example (continued): Show that \( f(x) = \cos(x^2) - 1 = -\frac{x^4}{2} + O(x^8) \).
Use this result to evaluate

\[
\lim_{x \to 0} \frac{\cos(x^2) - 1}{x^4}.
\]

Solution (continued):
To evaluate

\[
\lim_{x \to 0} \frac{\cos(x^2) - 1}{x^4}
\]
we use the Arithmetic Rules to get

\[
\lim_{x \to 0} \frac{\cos(x^2) - 1}{x^4} = \lim_{x \to 0} \frac{-\frac{x^4}{2} + O(x^8)}{x^4}
\]

\[
= \lim_{x \to 0} \frac{-\frac{1}{2} + O(x^4)}{x^4}
\]

\[
= -\frac{1}{2}
\]

since

\[
\lim_{x \to 0} O(x^n) = 0
\]
for every \( n > 0 \).
Big-O Arithmetic

**Example:** Let \( f(x) = \sin(x)(e^{-x^2} - 1) \). Show that

\[
f(x) = -x^3 + O(x^5).
\]

**Solution:** We know that

\[
\sin(x) = x + O(x^3)
\]

and that

\[
e^u = 1+u+O(u^2) \Rightarrow e^u - 1 = u+O(u^2) \Rightarrow e^{-x^2} - 1 = -x^2 + O(x^4)
\]

since

\[
O((-x^2)^2) = O(x^4).
\]

Therefore using the Arithmetic Rules for Big-O:

\[
\begin{align*}
\sin(x)(e^{-x^2} - 1) &= (x + O(x^3))(-x^2 + O(x^4)) \\
&= -x^3 + xO(x^4) + (-x^2)O(x^3) + O(x^3)O(x^4) \\
&= -x^3 + O(x^5) + O(x^5) + O(x^7) \\
&= -x^3 + O(x^5).
\end{align*}
\]
Important Remark: Suppose

\[ f(x) = 1 - x^2 + O(x^4) \]

and

\[ g(x) = x + O(x^2). \]

Then

\[
\begin{align*}
f(x)g(x) &= (1 - x^2 + O(x^4))(x + O(x^2)) \\
&= 1 \cdot x + 1 \cdot O(x^2) - x^2 \cdot x - x^2 \cdot O(x^2) + x \cdot O(x^4) + O(x^4) \cdot O(x^2) \\
&= x + O(x^2) - x^3 + O(x^4) + O(x^5) + O(x^6) \\
&= x + O(x^2) + O(x^3) + O(x^4) + O(x^5) + O(x^6) \\
&= x + O(x^2)
\end{align*}
\]

since once we have the term \( O(x^2) \), all higher degree terms (such as \( x^3 \)) do not add any additional accuracy to the estimate.