L’Hôpital’s Rule

Created by

Barbara Forrest and Brian Forrest
L’Hôpital’s Rule

Recall: If \( h(x) = \frac{f(x)}{g(x)} \) and if

\[
\lim_{x \to a} f(x) = 0 = \lim_{x \to a} g(x),
\]

then we do not know whether or not \( \lim_{x \to a} h(x) \) exists.

For this reason, we call such a situation an *indeterminate form of type \( \frac{0}{0} \).*

Similarly, if

\[
\lim_{x \to a} f(x) = \pm \infty = \lim_{x \to a} g(x),
\]

we would not be able to determine immediately if the limit of the quotient exists.

We call this situation an *indeterminate form of type \( \frac{\infty}{\infty} \).*

**Note:** *L’Hôpital’s Rule* gives us a means to evaluate such limits.
L’Hôpital’s Rule

**Observation:** Let \( h(x) = \frac{f(x)}{g(x)} \) and
\[
\lim_{x \to a} f(x) = 0 = \lim_{x \to a} g(x).
\]
Assume that \( f(x) \) and \( g(x) \) have continuous derivatives with \( g'(a) \neq 0 \).
We know that for \( x \) near \( a \) we have that
\[
\frac{f(x)}{g(x)} \approx \frac{f(a) + f'(a)(x - a)}{g(a) + g'(a)(x - a)} = \frac{f'(a)}{g'(a)}
\]
since \( f(a) = 0 = g(a) \).
This might lead us to guess that if \( \lim_{x \to a} \frac{f(x)}{g(x)} \) exists, then
\[
\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{f'(a)}{g'(a)}.
\]

Moreover, since \( f'(x) \) and \( g'(x) \) are continuous with \( g'(a) \neq 0 \), we also have
\[
\frac{f'(a)}{g'(a)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}.
\]
Combining (*) and (**) gives us
\[
\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}.
\]
L’Hospital’s Rule

**Theorem: [ L’Hôpital’s Rule]**

Assume that $f'(x)$ and $g'(x)$ exist near $x = a$, $g'(x) \neq 0$ near $x = a$ except possibly at $x = a$, and that $\lim_{x \to a} \frac{f(x)}{g(x)}$ is an indeterminate form of type $\frac{0}{0}$ or $\frac{\infty}{\infty}$. Then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

provided that the latter limit exists (or is $\infty$ or $-\infty$).

Moreover, this rule remains valid for one-sided limits and for limits at $\pm \infty$.

**Note:** The proof of L’Hôpital’s Rule uses an upgraded version of the MVT.
L’Hôpital’s Rule

Example: Evaluate

\[
\lim_{x \to 0} \frac{e^x - 1}{x}.
\]

Solution: Let \( f(x) = e^x - 1 \) and \( g(x) = x \). Then

\[
\lim_{x \to 0} e^x - 1 = e^0 - 1 = 0 = \lim_{x \to 0} x \Rightarrow \text{type } \frac{0}{0}.
\]

Since \( f'(x) = e^x \) and \( g'(x) = 1 \), by L’Hôpital’s Rule

\[
\lim_{x \to 0} \frac{e^x - 1}{x} = \lim_{x \to 0} \frac{e^x}{1} = 1.
\]

Note:

\[
\lim_{x \to 0} \frac{e^x - 1}{x}
\]

is the derivative of \( f(x) = e^x \) at \( x = 0 \).
**L’Hôpital’s Rule**

**Example:** Evaluate

\[
\lim_{x \to 0} \frac{e^x - 1 - x}{x^2}.
\]

**Solution:** Let \( f(x) = e^x - 1 - x \) and \( g(x) = x^2 \). Then

\[
\lim_{x \to 0} e^x - 1 - x = 0 = \lim_{x \to 0} x^2 \Rightarrow \text{type } \frac{0}{0}.
\]

Then \( f'(x) = e^x - 1 \) and \( g'(x) = 2x \) and

\[
\lim_{x \to 0} e^x - 1 = 0 = \lim_{x \to 0} 2x \Rightarrow \lim_{x \to 0} \frac{f'(x)}{g'(x)} \text{ is type } \frac{0}{0}.
\]

Let \( F(x) = f'(x) = e^x - 1 \) and \( G(x) = g'(x) = 2x \), then \( F'(x) = e^x \) and \( G'(x) = 2 \) so

\[
\lim_{x \to 0} \frac{e^x - 1 - x}{x^2} = \lim_{x \to 0} \frac{f'(x)}{g'(x)} = \lim_{x \to 0} \frac{F(x)}{G(x)} = \lim_{x \to 0} \frac{e^x}{2} = \frac{1}{2}.
\]
**L’Hôpital’s Rule**

**Example:** Evaluate

\[ \lim_{x \to 0^+} \frac{e^x - 1}{x^2}. \]

**Solution:** Let \( f(x) = e^x - 1 \) and \( g(x) = x^2 \). Then \( f'(x) = e^x \) and \( g'(x) = 2x \), so we have

\[ \lim_{x \to 0^+} \frac{e^x - 1}{x^2} = \lim_{x \to 0^+} \frac{e^x}{2x}. \]

But

\[ \lim_{x \to 0^+} 2x = 0. \]

so by applying L’Hôpital’s Rule again we get

\[ \lim_{x \to 0^+} \frac{e^x - 1}{x^2} = \lim_{x \to 0^+} \frac{e^x}{2x} = \frac{1}{2}. \]

**Warning:** This is wrong since \( \lim_{x \to 0^+} e^x = 1 \neq 0 \) so

\[ \lim_{x \to 0^+} \frac{e^x - 1}{x^2} = \lim_{x \to 0^+} \frac{e^x}{2x} = \infty. \]
L’Hôpital’s Rule

Example: Evaluate \( \lim_{x \to \infty} \frac{\ln(x)}{x} \).

Solution: Let \( f(x) = \ln(x) \) and \( g(x) = x \). Then

\[
\lim_{x \to \infty} f(x) = \infty = \lim_{x \to \infty} g(x),
\]

so this is an indeterminate form of the type \( \frac{\infty}{\infty} \).

Differentiating \( f(x) \) and \( g(x) \) gives us \( f'(x) = \frac{1}{x} \) and \( g'(x) = 1 \). Therefore,

\[
\lim_{x \to \infty} \frac{f'(x)}{g'(x)} = \lim_{x \to \infty} \frac{\frac{1}{x}}{1} = \lim_{x \to \infty} \frac{1}{x} = 0.
\]

Thus \( \lim_{x \to \infty} \frac{\ln(x)}{x} = 0 \).
L’Hôpital’s Rule

**Remark:** Up until now we have dealt with two types of indeterminate forms which we have denoted by $\frac{0}{0}$ and $\frac{\infty}{\infty}$. There are five more standard indeterminate forms which we will denote by

$$0 \cdot \infty, \quad \infty - \infty, \quad 1^\infty, \quad \infty^0, \quad \text{and} \quad 0^0.$$  

For example, an indeterminate form of type $0 \cdot \infty$ arises from the function $h(x) = f(x)g(x)$ when

$$\lim_{x \to a} f(x) = 0$$

and

$$\lim_{x \to a} g(x) = \infty.$$  

Similarly, the function $(g(x))^f(x)$ would produce an indeterminate form of type $\infty^0$.

**Note:** All of the above forms can be converted to forms of the type $\frac{0}{0}$ or $\frac{\infty}{\infty}$. 
**L’Hôpital’s Rule**

**Example:** Evaluate

\[
\lim_{{x \to 0^+}} x \ln(x).
\]

**Solution:** This is an indeterminate form of type \(0 \cdot \infty\) since

\[
\lim_{{x \to 0^+}} x = 0
\]

and

\[
\lim_{{x \to 0^+}} \ln(x) = -\infty.
\]

We can rewrite this example as

\[
\lim_{{x \to 0^+}} \frac{\ln(x)}{\frac{1}{x}}
\]

which is type \(\frac{\infty}{\infty}\). L’Hôpital’s Rule gives us that

\[
\lim_{{x \to 0^+}} x \ln(x) = \lim_{{x \to 0^+}} \frac{\ln(x)}{\frac{1}{x}}
\]

\[
= \lim_{{x \to 0^+}} \left( \frac{1}{x} \right)
\]

\[
= 0.
\]
L’Hôpital’s Rule

**Example:** Evaluate

\[
\lim_{x \to \infty} \left(1 + \frac{1}{x}\right)^x.
\]

**Solution:** This is type \(1^\infty\). We write

\[
\left(1 + \frac{1}{x}\right)^x = e^{\ln\left((1 + \frac{1}{x})^x\right)} = e^{x \ln\left(1 + \frac{1}{x}\right)}.
\]

Since \(x \ln \left(1 + \frac{1}{x}\right)\) is type \(0 \cdot \infty\), we rewrite this as

\[
\frac{\ln\left(1 + \frac{1}{x}\right)}{\frac{1}{x}}
\]

which is type \(\frac{0}{0}\). L’Hôpital’s Rule gives us that

\[
\lim_{x \to \infty} \frac{\ln\left(1 + \frac{1}{x}\right)}{\frac{1}{x}} = \lim_{x \to \infty} \frac{\frac{-1}{(1 + \frac{1}{x})}}{\frac{-1}{x^2}} = \lim_{x \to \infty} \frac{1}{1 + \frac{1}{x}} = \frac{1}{1 + \frac{1}{x}} = 1.
\]

So

\[
\lim_{x \to \infty} \left(1 + \frac{1}{x}\right)^x = e^{\lim_{x \to \infty} \left(\ln\left(1 + \frac{1}{x}\right)^x\right)} = e.
\]
L’Hôpital’s Rule

Problem:

Show that

\[
\lim_{x \to 0} \frac{4(e^{x^3} - 1 - x^3 - \frac{x^6}{2})^2}{x^6 \tan(x^7) \sin(2x^5)} = \frac{1}{18}.
\]