Applications of the MVT: Increasing Function Theorem

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Functions

Definition: [Increasing and Decreasing Functions]

Suppose that \( f(x) \) is defined on an interval \( I \).

i) We say that \( f(x) \) is increasing on \( I \) if \( f(x_1) < f(x_2) \) for all \( x_1, x_2 \in I \) with \( x_1 < x_2 \).

ii) We say that \( f(x) \) is decreasing on \( I \) if \( f(x_1) > f(x_2) \) for all \( x_1, x_2 \in I \) with \( x_1 < x_2 \).

iii) We say that \( f(x) \) is non-decreasing on \( I \) if \( f(x_1) \leq f(x_2) \) for all \( x_1, x_2 \in I \) with \( x_1 < x_2 \).

iv) We say that \( f(x) \) is non-increasing on \( I \) if \( f(x_1) \geq f(x_2) \) for all \( x_1, x_2 \in I \) with \( x_1 < x_2 \).

Such functions are said to be monotonically increasing or decreasing on \( I \).

**Question:** How can we determine if a function \( f(x) \) is either increasing or decreasing on an interval \( I \)?
Increasing Function Theorem

**Observation:** Assume that

\[ f(x) = mx + b. \]

If \( m > 0 \), the graph of the function slopes upward as we move from left to right. In other words, if \( x_1 < x_2 \), then

\[ f(x_1) = mx_1 + b < mx_2 + b = f(x_2). \]

**Note:** \( f'(x) = m > 0 \) for all \( x \in \mathbb{R} \).
Increasing Function Theorem

$f'(x) > 0$

$f(x)$ always increasing?

**Question:** If $f(x)$ is such that $f'(x) > 0$ for all $x \in I$, is $f(x)$ increasing on $I$?
Theorem: [The Increasing/Decreasing Function Theorem]

i) Let $I$ be an interval and assume that $f'(x) > 0$ for all $x \in I$. If $x_1 < x_2$ are two points in $I$, then

$$f(x_1) < f(x_2).$$

That is, $f(x)$ is increasing on $I$.

ii) Let $I$ be an interval and assume that $f'(x) \geq 0$ for all $x \in I$. If $x_1 < x_2$ are two points in $I$, then

$$f(x_1) \leq f(x_2).$$

That is, $f(x)$ is non-decreasing on $I$.

iii) Let $I$ be an interval and assume that $f'(x) < 0$ for all $x \in I$. If $x_1 < x_2$ are two points in $I$, then

$$f(x_1) > f(x_2).$$

That is, $f(x)$ is decreasing on $I$.

iv) Let $I$ be an interval and assume that $f'(x) \leq 0$ for all $x \in I$. If $x_1 < x_2$ are two points in $I$, then

$$f(x_1) \geq f(x_2).$$

That is, $f(x)$ is non-increasing on $I$. 
Increasing Function Theorem

i) Let $I$ be an interval and assume that $f'(x) > 0$ for all $x \in I$. If $x_1 < x_2$ are two points in $I$, then

$$f(x_1) < f(x_2).$$
Increasing Function Theorem

**Proof of i):** Assume that \( f'(x) > 0 \) for all \( x \in I \). Let \( x_1, x_2 \in I \) with \( x_1 < x_2 \). By the MVT there exists \( c \in (x_1, x_2) \) with

\[
\frac{f(x_2) - f(x_1)}{x_2 - x_1} = f'(c) > 0.
\]

Since \( x_2 - x_1 > 0 \), we have

\[
f(x_2) - f(x_1) > 0
\]

and hence that

\[
f(x_2) > f(x_1).
\]
Increasing Function Theorem

**Question:** If $f(x)$ is increasing on an interval $I$ and differentiable on $I$, then must $f'(x) > 0$ for all $x \in I$?

**Solution:** Let $f(x) = x^3$. Since $f'(x) = 3x^2$, we have

$$f'(0) = 0$$

but $f(x)$ is increasing on all of $\mathbb{R}$. 
Question:

1) Is the function $f(x) = x^3$ increasing on $[0, 1]$? **Yes!**

2) If $f(x)$ is everywhere differentiable and if $f'(c) > 0$, does this mean that there is an open interval $(a, b)$ containing $c$ on which $f(x)$ is increasing? **No!**