Applications of the MVT: Concavity

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**Concavity**

**Question:** If $f''(x) > 0$ on some interval $I$, what can we say about the shape of the graph of the function $f(x)$?

**Observation:** $f'(x)$ is increasing so the tangent lines to $f(x)$ rotate counter-clockwise $\Rightarrow f(x)$ is **concave upwards**.
Concavity

Question: What if \( f''(x) < 0 \) on \( I \)?

Observation: \( f'(x) \) is decreasing so tangent lines to \( f(x) \) rotate clockwise \( \Rightarrow f(x) \) is concave downwards.
Question: How can we quantify *concave upwards*?

Observation: The secant line lies **above** the graph of $f(x)$. 
Definition: [Concave Upwards]
The graph of \( f(x) \) is concave upwards on an interval \( I \) if for every pair of points \( a \) and \( b \) in \( I \), the secant line joining \( (a, f(a)) \) and \( (b, f(b)) \) lies above the graph of \( f(x) \).
**Question:** How can we quantify *concave downwards*?

**Observation:** The secant line lies below the graph of $f(x)$. 
Concave Downwards

The graph of $f(x)$ is concave downwards on an interval $I$ if for every pair of points $a$ and $b$ in $I$, the secant line joining $(a, f(a))$ and $(b, f(b))$ lies below the graph of $f(x)$.
Second Derivative Test for Concavity

**Theorem: [Second Derivative Test for Concavity]**

i) If \( f''(x) > 0 \) for each \( x \) in an interval \( I \), then the graph of \( f(x) \) is concave upwards on \( I \).

ii) If \( f''(x) < 0 \) for each \( x \) in an interval \( I \), then the graph of \( f(x) \) is concave downwards on \( I \).
Inflection Points

Example:
1) \( f(x) \) is concave upwards on \([a, c]\).
2) \( f(x) \) is concave downwards on \([c, b]\).
3) \( f(x) \) changes its concavity at \( x = c \). The point \((c, f(c))\) is called an inflection point.
Inflection Points

Definition: [Inflection Point]

A point \( c \) is called an inflection point for the function \( f(x) \) if

i) \( f(x) \) is continuous at \( x = c \), and
ii) the concavity of \( f(x) \) changes at \( x = c \).

Important Note:

1) Typically an inflection point \( x = c \) would occur when the second derivative changes from positive to negative, or vice versa.
2) If \( f''(x) \) is continuous, the Intermediate Value Theorem requires that \( f''(c) = 0 \).
**Theorem: [Test for Inflection Points]**

If $f''(x)$ is continuous at $x = c$ and $(c, f(c))$ is an inflection point for $f(x)$, then

$$f''(c) = 0.$$  

**WARNING:** This theorem shows us how to locate candidates for inflection points. However, $f''(c) = 0$ does not mean that an inflection point always occurs when $x = c$. 
Inflection Points

Example: Let $f(x) = x^3$. Then $f''(x) = 6x$.

1) $f''(x) < 0$ if $x < 0$ so $f(x)$ is concave downwards on $(-\infty, 0]$.
2) $f''(x) > 0$ if $x > 0$ so $f(x)$ is concave upwards on $[0, \infty)$.
3) The function has a point of inflection at $x = 0$. 
Inflection Points

Example: Let \( f(x) = x^4 \). Then \( f''(x) = 12x^2 \).

1) \( f''(x) > 0 \) for all \( x \neq 0 \) so \( f(x) \) is concave upwards on \( \mathbb{R} \).
2) \( f''(0) = 0 \) but the function does \textit{not} have a point of inflection at \( x = 0 \).
Inflection Points

Example:
The diagram represents the graph of the population $P$ of a particular bacteria over time $t$ in a restricted environment.

**Important Fact:** The point of inflection occurs at the time $t_I$ when the growth rate of the bacteria is at its highest value.