Local Extrema

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Global Extrema

Definition: [Global Extrema]

Suppose that $f(x)$ is defined on some set $S$.

1) We say that $f(x)$ has a global maxima on $S$ at $x = c$ if

$$f(x) \leq f(c)$$

for every $x \in S$.

2) We say that $f(x)$ has a global minimum on $S$ at $x = c$ if

$$f(c) \leq f(x)$$

for every $x \in S$. 
Recall: The *Extreme Value Theorem* tells us that if $f(x)$ is continuous on $[a, b]$, then there exists $c, d \in [a, b]$ such that

$$f(c) \leq f(x) \leq f(d)$$

for all $x \in [a, b]$, and each of $c$ and $d$ is either

1) an endpoint, or
2) is in the open interval $(a, b)$. 

Global Extrema
Local Extrema

Definition: [Local Extrema]

1) We say that \( f(x) \) has a *local maximum* at \( x = c \) if there exists an open interval \( (a, b) \) containing \( c \) such that

\[
f(x) \leq f(c)
\]

for every \( x \in (a, b) \).

2) We say that \( f(x) \) has a *local minimum* at \( x = c \) if there exists an open interval \( (a, b) \) containing \( c \) such that

\[
f(c) \leq f(x)
\]

for every \( x \in (a, b) \).
Local and Global Extrema

$y = g(x)$ is defined on $[a, f]$. 

- **Local and Global Minimum**: $g(x)$ reaches its minimum at $x = a$ and $x = f$.
- **Local Maximum**: $g(x)$ reaches its maximum at $x = b$ and $x = f$.
- **Local Minimum**: $g(x)$ reaches its minimum at $x = c$, $x = d$, and $x = e$. The global maximum occurs at $x = f$. 

**Notes**: 
- The function $g(x)$ changes behavior at $x = b$ and $x = f$. 
- The interval $[a, f]$ is where the function $g(x)$ is defined.
Local Extrema

**Problem:**

If $f$ has either a local maximum or local minimum at $x = c$ and $f$ is differentiable at $x = c$, what can we say about $f'(c)$?
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Solution: Assume that $f(x)$ has a local maximum at $x = c$ and that $c \in (a, b)$ with $f(x) \leq f(c)$ for all $x \in (a, b)$. Assume that $f'(c)$ exists.

1. If $a < c + h < c$, then
   \[
   \frac{f(c+h)-f(c)}{h} \geq 0 \Rightarrow f'(c) = \lim_{h \to 0^-} \frac{f(c+h)-f(c)}{h} \geq 0.
   \]

2. If $c < c + h < b$, then
   \[
   \frac{f(c+h)-f(c)}{h} \leq 0 \Rightarrow f'(c) = \lim_{h \to 0^+} \frac{f(c+h)-f(c)}{h} \leq 0.
   \]

Hence $f'(c) = 0$. 
Local Extrema

**Theorem: [Local Extrema Theorem]**

Assume that $f(x)$ has either a local maximum or a local minimum at $x = c$. If $f(x)$ is differentiable at $x = c$, then

$$f'(c) = 0.$$  

**Question:**

If $f'(c) = 0$ must $x = c$ be either a local maximum or local minimum?
Local Extrema

Example: Let \( f(x) = x^3 \). Then \( f'(x) = 3x^2 \) so that
\[
f'(0) = 0.
\]
But \( f(x) \) has neither a local maximum nor a local minimum at \( x = c \) since \( f(x) \) is always increasing.
Local Extrema

\[ f(x) = |x| \]

**Question:** If \( f(x) \) has a local maximum or minimum at \( x = c \), must \( f'(c) = 0 \)?

**Example:** Let \( f(x) = |x| \). Then \( f(x) \) has a local (and global) minimum at \( x = 0 \), but \( f'(0) \) does not exist.
Definition: [Critical Point]

A point $c$ in the domain of a function $f(x)$ is called a critical point for $f(x)$ if either

$$ f'(c) = 0 $$

or

$$ f'(c) \text{ does not exist.} $$
Theorem: [Extreme Value Theorem (EVT)]

Assume that $f(x)$ is continuous on $[a, b]$. Then there exists $c, d \in [a, b]$ such that

$$f(c) \leq f(x) \leq f(d)$$

for every $x \in [a, b]$.

Question: How can we find $c$ and $d$?
Global Extrema and the EVT

Two Cases:

1) $d$ is an endpoint.

2) $d \in (a, b) \Rightarrow d$ is a local max $\Rightarrow d$ is a critical point.

Similarly for $c$. 

Global Extrema and the EVT

Two Cases:

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Global Extrema and the EVT

Two Cases:
1) \( d \) is an endpoint.

2) \( d \in (a, b) \Rightarrow d \) is a local max \( \Rightarrow d \) is a critical point.

Similarly for \( c \).
Finding Local Extrema

Question:
How do we know if a critical point \( x = c \) is actually a local maximum or a local minimum?
Question:

How do we know if a critical point $x = c$ is actually a local maximum or a local minimum?