Linear Approximation: Basics

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**Question:** Does there exist a geometric interpretation of the derivative?

**Definition:** [Tangent Line]

The *tangent line to the graph of* \( f(x) \) *at* \( x = a \) *is the line passing through* \((a, f(a))\) *with slope equal to* \( f'(a) \). That is

\[
y = f(a) + f'(a)(x - a).
\]
**Linear Approximation**

**Fundamental Observation:**

Suppose that $f(x)$ is differentiable at $x = a$ with derivative $f'(a)$. Then

$$\lim_{x \to a} \frac{f(x) - f(a)}{x - a} = f'(a).$$

Hence for values of $x$ that are close to $a$ we have

$$f(x) - f(a) \approx f'(a)(x - a). \quad (*)$$

Rearranging $(*)$, we get

$$f(x) - f(a) \approx f'(a)(x - a)$$

and finally that

$$f(x) \approx f(a) + f'(a)(x - a). \quad (**)$$
Definition: [Linear Approximation]

Let \( y = f(x) \) be differentiable at \( x = a \). The linear approximation to \( f(x) \) at \( x = a \) is the function

\[
L_a^f(x) = f(a) + f'(a)(x - a).
\]

\( L_a^f(x) \) is also called the **linearization** of \( f(x) \) or the **tangent line approximation** to \( f(x) \) at \( x = a \).

**Note:** If \( f(x) \) is clear from the context, then we will simply write \( L_a(x) \).
**Linear Approximation**

\[ y = L_a^f(x) = f(a) + f'(a)(x - a) \]

**Summary:** If
\[ L_a^f(x) = f(a) + f'(a)(x - a) \]
then if \( x \simeq a \),
\[ L_a^f(x) \simeq f(x). \]

**Observation:** The graph of \( L_a^f(x) \) is the tangent line to the graph of \( f(x) \) through \((a, f(a))\).
Fundamental Properties of $L^f_a(x)$

Three Fundamental Properties of $L^f_a(x)$:

Assume that $f(x)$ is differentiable at $x = a$ with

$$L^f_a(x) = f(a) + f'(a)(x - a).$$

Then:

1) $L^f_a(a) = f(a)$.

2) $L^f_a(x)$ is differentiable at $x = a$ and $L^f_a'(a) = f'(a)$.

3) $L^f_a(x)$ is the **only** first degree polynomial with Property (1) and Property (2).
Recall: The Fundamental Trig Limit

\[
\lim_{x \to 0} \frac{\sin(x)}{x} = 1.
\]

Observe: If \( f(x) = \sin(x) \), then \( f(0) = 0 \) and \( f'(0) = \cos(0) = 1 \)
so

\[
\sin(x) \approx L_0(x) = x
\]
when \( x \approx 0 \).
**Example:** Use linear approximation to estimate $\sin(0.01)$.

**Solution:** We have

$$\sin(0.01) \approx L_0(0.01) = x \mid_{0.01} = 0.01$$

**Note:** In fact

$$\sin(0.01) = 0.00999983$$

to eight decimal places.

So the error is

Error \quad = \quad |\sin(0.01) - L_0(0.01)|

\[\approx 0.00000017\]

\[= 1.7 \times 10^{-7}\]
Example: Let \( f(x) = \tan(\sqrt{x}) \) and \( a = 1 \).

Key Observation:
Over very small intervals differentiable functions appear like lines.