Linear Approximation: Applications

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**Problem:** Assume that we know the value of a function $f(x)$ at a point $a$. How do we estimate the change we could expect in the value of $f(x)$ if we move to a point $x_1$ near $a$?

That is, we want to estimate the value of

$$\Delta y = f(x_1) - f(a)$$

if we change our variable by

$$\Delta x = x_1 - a$$
Estimating Change

\[ f'(a) \Delta x \]

**Solution:** If we were to use linear approximation, we get that

\[
\Delta y = f(x_1) - f(a) \\
\approx L_a(x_1) - f(a) \\
= (f(a) + f'(a)(x_1 - a)) - f(a) \\
= f'(a)(x_1 - a) \\
= f'(a) \Delta x.
\]

That is,

\[
\Delta y \approx f'(a) \Delta x.
\]
Example: A metal sphere of radius 10 cm expands when heated so that its radius increases by 0.01 cm. Estimate the change in the volume of the sphere.

Solution: We know that the volume ($V$) of the sphere with radius $r$ is given by

$$V = V(r) = \frac{4}{3}\pi r^3$$

and that $V'(r) = 4\pi r^2$. 
Solution (continued): Our focal point is at $r = 10$, so 
\[ V'(10) = 400\pi. \]
We also know that $\Delta r = .01$, so 
\[ \Delta V = V(10.01) - V(10) \approx V'(10)\Delta r \]
\[ = 400\pi(.01) \]
\[ = 4\pi \text{ cm}^3. \]
Qualitative Analysis of Functions

**Problem:** How does the function $f(x) = e^{-x^2}$ behave near $x = 0$?

**Solution:**

**Step 1:** Start with the simpler function $h(u) = e^u$.

Since

$$h(0) = h'(0) = e^0 = 1$$

we get that

$$e^u \approx L_0^h(u) = 1 + u$$

so long as $u$ is near 0.
Problem: How does the function
\[ f(x) = e^{-x^2} \]
behave near \( x = 0 \)?

Solution (continued):

Step 2: We know that
\[ e^u \approx L^h_0(u) = 1 + u \]
so long as \( u \) is near 0.
**Problem:** How does the function 

\[ f(x) = e^{-x^2} \]

behave near \( x = 0 \)?

**Solution (continued):**

**Step 2:** We know that

\[ e^u \approx L_0^h(u) = 1 + u \]

so long as \( u \) is near 0.

If \( x \) is close to 0, then so is \( u = -x^2 \). Letting \( u = -x^2 \), we get

\[ y = e^{-x^2} \approx 1 + (-x^2) = 1 - x^2 \]

if \( x \approx 0 \).