Implicit Differentiation

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Example: Consider the relation

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Note: This relation is not a function. However, by restricting the range of the relation we can generate many functions. We say these functions are \textit{implicitly} defined by the relation.
Relations and Implicit Functions

\[ x^2 + y^2 = 1 \]

\[ y = +\sqrt{1 - x^2} = f(x) \]

\[ y = -\sqrt{1 - x^2} = g(x) \]

Example (continued): The relation

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implies two functions

\[ y = f(x) = +\sqrt{1 - x^2} \quad \text{and} \quad y = g(x) = -\sqrt{1 - x^2} \]
Relations and Implicit Functions

\[ y = f(x) = +\sqrt{1 - x^2} \]

\[ m = \frac{dy}{dx} = -\frac{x}{y} \]

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with derivatives

\[ f'(x) = -\frac{x}{\sqrt{1 - x^2}} = -\frac{x}{y} \]
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Implicit Differentiation

Example (continued): If

\[ y = h(x) \]

is any differentiable function which satisfies

\[ x^2 + y^2 = 1 \]

then

\[ \frac{d}{dx}(x^2 + y^2) = \frac{d}{dx}(1). \]

Hence

\[ 2x + 2y \cdot \frac{dy}{dx} = 0. \]

Finally

\[ 2y \cdot \frac{dy}{dx} = -2x \Rightarrow \frac{dy}{dx} = -\frac{x}{y} \]

provided that \( y \neq 0 \).

Note: The process of finding the derivative without knowing the explicit formula for the function is called *implicit differentiation*. 
Example: The graph of the relation
\[ \{(x, y) \mid x^3 + y^3 = 6xy\} \]
is called the Folium of Descartes.
Folium of Descartes

$y = f(x)$

portion of $x^3 + y^3 = 6xy$

**Example:** The graph of the relation

$$\{(x, y)| x^3 + y^3 = 6xy\}$$

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Example: The graph of the relation
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is called the Folium of Descartes.
Example: Find \( h'(3) \) if \( y = h(x) \) is a differentiable function with \( h(3) = 3 \) satisfying
\[
x^3 + y^3 = 6xy.
\]

Then
\[
\frac{d}{dx}(x^3 + y^3) = \frac{d}{dx}(6xy).
\]

Hence
\[
3x^2 + 3y^2 \frac{dy}{dx} = 6y + 6x \frac{dy}{dx}
\]

and
\[
\frac{dy}{dx} = \frac{6y - 3x^2}{3y^2 - 6x}.
\]

Letting \( x = 3 \) and \( y = 3 \) gives
\[
h'(3) = \frac{dy}{dx}\bigg|_{(3,3)} = \frac{6(3) - 3(3^2)}{3(3^2) - 6(3)} = -1.
\]
Example: Suppose that

\[ x^4 + y^4 = -1 - x^2 y^2. \] \(^(*)\)

Find \( \frac{dy}{dx} \).

Solution: We get

\[ \frac{d}{dx}(x^4 + y^4) = \frac{d}{dx}(-1 - x^2 y^2) \]

so

\[ \frac{dy}{dx} = \frac{-2xy^2 - 4x^3}{4y^3 + 2x^2 y}. \] \(^{(**)}\)

Important Observation:
The equation \((*)\) has no solutions, so \((**)\) is meaningless!