Derivatives

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Average Rate of Change

**Definition: [Average Rate of Change]**
Given a function $f(t)$, we can define the average rate of change of $f(t)$ as $t$ goes from $t_0$ to $t_1$ to be the ratio

$$
\frac{f(t_1) - f(t_0)}{t_1 - t_0}.
$$

**Note:** If we fix a point $t_0$ and let $h = t_1 - t_0$ be small, then

$$
\frac{f(t_0 + h) - f(t_0)}{h}
$$

is an estimate of the instantaneous rate of change of $f(t)$ at $t_0$.

**Definition: [Instantaneous Rate of Change]**
Given a function $f(t)$, we can define the instantaneous rate of change of $f(t)$ at $t_0$ to be

$$
\lim_{h \to 0} \frac{f(t_0 + h) - f(t_0)}{h}.
$$
The Derivative

**Definition: [Newton Quotient]**
Given a function \( f(t) \), a point \( t = a \) and \( h \neq 0 \), the ratio

\[
\frac{f(a + h) - f(a)}{h}
\]

is called a *Newton Quotient* for \( f(t) \) centered at \( t = a \).

**Note:** Geometrically, the Newton Quotient represents the slope of the secant line to the graph of \( f(t) \) through the points \( (a, f(a)) \) and \( (a + h, f(a + h)) \).
**The Derivative**

**Definition: [The Derivative at $t = a$]**
We say that the function $f(t)$ is differentiable at $t = a$ if

$$
\lim_{h \to 0} \frac{f(a + h) - f(a)}{h}
$$

exists.

In this case, we write

$$
f'(a) = \lim_{h \to 0} \frac{f(a + h) - f(a)}{h}
$$

and $f'(a)$ is the derivative of $f(t)$ at $t = a$.

**Note:** If $t = a + h$, then as $h \to 0$ we have $t \to a$. Furthermore, since $h = t - a$, we get

$$
f'(a) = \lim_{h \to 0} \frac{f(a + h) - f(a)}{h}
= \lim_{t \to a} \frac{f(t) - f(a)}{t - a}
$$

provided the limits exist.
Derivative as Instantaneous Rate of Change

**Observation:** If \( y = f(t) \), \( \triangle y = f(t) - f(a) \) and \( \triangle t = t - a \), then the Newton Quotient

\[
\frac{\triangle y}{\triangle t} = \frac{f(t) - f(a)}{t - a}
\]

is the *average* change in \( y \).

Therefore,

\[
f'(a) = \lim_{\triangle t \to 0} \frac{\triangle y}{\triangle t}
\]

is the limit of average rates of change over smaller and smaller intervals and so this represents the *instantaneous rate of change of \( y \) with respect to \( t \).*
Derivative as Instantaneous Rate of Change

Example:
Let \( s = s(t) \) represent the displacement of an object. Then

\[
s'(t_0) = \lim_{\Delta t \to 0} \frac{\Delta s}{\Delta t} = \lim_{t \to t_0} \frac{s(t) - s(t_0)}{t - t_0} = \lim_{h \to 0} \frac{s(t_0 + h) - s(t_0)}{h} = v(t_0)
\]

where \( v(t_0) \) is the instantaneous velocity of the object at time \( t_0 \).

Fact: Velocity is the derivative of displacement.
The Tangent Line

**Question:** Does there exist a geometric interpretation of the derivative?

**Observation:** If \( f(x) \) is differentiable at \( x = a \), the slopes of the secant lines through \((a, f(a))\) and \((a + h, f(a + h))\) converge to \( f'(a) \) as \( h \to 0 \).

**Definition: [Tangent Line]**

The **tangent line to the graph of** \( f(x) \) **at** \( x = a \) **is the line passing through** \((a, f(a))\) **with slope equal to** \( f'(a) \). That is,

\[
y = f(a) + f'(a)(x - a).
\]
Derivative of a Constant Function

$f(x) = c$

$f'(a) = \text{slope} = 0$

Example: [Derivative of a Constant Function]

Assume that $f(x) = c$ for all $x \in \mathbb{R}$ and let $a \in \mathbb{R}$. Then

\[
 f'(a) = \lim_{h \to 0} \frac{f(a + h) - f(a)}{h}
 = 0
\]
Example: [Derivative of a Linear Function]

Assume that

\[ f(x) = mx + b \]

for all \( x \in \mathbb{R} \). Then

\[
\begin{align*}
  f'(a) & = \lim_{h \to 0} \frac{f(a + h) - f(a)}{h} \\
  & = \lim_{h \to 0} \frac{(ma + mh + b) - (ma + b)}{h} \\
  & = \lim_{h \to 0} \frac{mh}{h} \\
  & = m.
\end{align*}
\]