Arithmetic Rules for Differentiation

Created by

Barbara Forrest and Brian Forrest
Theorem: [The Arithmetic Rules for Differentiation]

Assume that \( f(x) \) and \( g(x) \) are both differentiable at \( x = a \).

1) **The Constant Multiple Rule:**
   Let \( h(x) = cf(x) \). Then \( h(x) \) is differentiable at \( x = a \) and
   \[
   h'(a) = c \cdot f'(a).
   \]

2) **The Sum Rule:**
   Let \( h(x) = f(x) + g(x) \). Then \( h(x) \) is differentiable at \( x = a \) and
   \[
   h'(a) = f'(a) + g'(a).
   \]

3) **The Product Rule:**
   Let \( h(x) = f(x)g(x) \). Then \( h(x) \) is differentiable at \( x = a \) and
   \[
   h'(a) = f'(a)g(a) + f(a)g'(a).
   \]
Theorem: [The Arithmetic Rules for Differentiation (continued)]

Assume that $f(x)$ and $g(x)$ are both differentiable at $x = a$.

4) **The Reciprocal Rule:**
   Let $h(x) = \frac{1}{f(x)}$. If $f(a) \neq 0$, then $h(x)$ is differentiable at $x = a$ and
   
   $$h'(a) = \frac{-f'(a)}{(f(a))^2}.$$

5) **The Quotient Rule:**
   Let $h(x) = \frac{f(x)}{g(x)}$. If $g(a) \neq 0$, then $h(x)$ is differentiable at $x = a$ and
   
   $$h'(a) = \frac{f'(a)g(a) - f(a)g'(a)}{(g(a))^2}.$$
1) Proof of the Constant Multiple Rule:

Assume that \( c \in \mathbb{R} \) and that \( f(x) \) is differentiable at \( x = a \). Then

\[
(cf)'(a) = \lim_{h \to 0} \frac{(cf)(a + h) - (cf)(a)}{h} = \lim_{h \to 0} \frac{c \cdot f(a + h) - c \cdot f(a)}{h} = c \cdot \left( \lim_{h \to 0} \frac{f(a + h) - f(a)}{h} \right) = c \cdot f'(a).
\]
2) **Proof of the Sum Rule:**

Assume that $f(x)$ and $g(x)$ are differentiable at $x = a$. Then

$$(f + g)'(a) = \lim_{h \to 0} \frac{(f + g)(a + h) - (f + g)(a)}{h}$$

$$= \lim_{h \to 0} \frac{f(a + h) + g(a + h) - f(a) - g(a)}{h}$$

$$= \lim_{h \to 0} \frac{f(a + h) - f(a)}{h} + \lim_{h \to 0} \frac{g(a + h) - g(a)}{h}$$

$$= f'(a) + g'(a).$$
3) **Proof of the Product Rule:**

**Observe:** We have

\[(fg)(a + h) - (fg)(a) = [f(a + h)g(a + h) - f(a + h)g(a)] + [f(a + h)g(a) - f(a)g(a)].\]

\[
(fg)'(a) = \lim_{h \to 0} \frac{(fg)(a + h) - (fg)(a)}{h} \\
= \lim_{h \to 0} \frac{f(a + h)(g(a + h) - g(a))}{h} \\
+ \lim_{h \to 0} \frac{g(a)(f(a + h) - f(a))}{h} \\
= \lim_{h \to 0} f(a + h) \cdot \lim_{h \to 0} \frac{(g(a + h) - g(a))}{h} \\
+ g(a) \cdot \lim_{h \to 0} \frac{(f(a + h) - f(a))}{h} \\
= f(a)g'(a) + f'(a)g(a).
\]
4) **Proof of the Reciprocal Rule:**

Assume that \( f(x) \) is differentiable at \( x = a \). Then

\[
\left( \frac{1}{f} \right)'(a) = \lim_{h \to 0} \frac{1}{f(a+h)} - \frac{1}{f(a)}
\]

\[
= \lim_{h \to 0} \frac{f(a) - f(a + h)}{f(a + h)f(a)h}
\]

\[
= - \lim_{h \to 0} \frac{f(a + h) - f(a)}{h} \cdot \lim_{h \to 0} \frac{1}{f(a + h)f(a)}
\]

\[
= -f'(a) \cdot \frac{1}{(f(a))^2} \quad \text{(by continuity at } x = a) \]

\[
= \frac{-f'(a)}{(f(a))^2}.
\]

5) **Proof of the Quotient Rule:**

The proof of the Quotient Rule is a combination of the Product Rule and the Reciprocal Rule.
Power Rule for Differentiation

**Note:** We have seen that

\[
\frac{d}{dx}(x) = 1 \quad \text{and} \quad \frac{d}{dx}(x^2) = 2x.
\]

Using the Binomial Theorem we can show that if \(n \in \mathbb{N}\), then

\[
\frac{d}{dx}(x^n) = nx^{n-1}.
\]

**Theorem: [The Power Rule for Differentiation]**

Assume that \(\alpha \in \mathbb{R}, \alpha \neq 0\), and \(f(x) = x^\alpha\). Then \(f(x)\) is differentiable and

\[
f'(x) = \alpha x^{\alpha-1}
\]

wherever \(x^{\alpha-1}\) is defined.
Differentiating Polynomials and Rational Functions

Examples: Differentiating Polynomials and Rational Functions

1) Let $P(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$ be a polynomial. Then $P(x)$ is always differentiable and

$$P'(x) = a_1 + 2a_2 x + 3a_3 x^2 + \cdots + na_n x^{n-1}.$$ 

2) Using the Quotient Rule, we see that a rational function

$$R(x) = \frac{P(x)}{Q(x)}$$

is differentiable at any point where $Q(x) \neq 0$. 
Differentiating Polynomials and Rational Functions

Example:

If

\[ R(x) = \frac{x + 2}{x^2 - 1}, \]

then \( R(x) \) is differentiable provided that \( x^2 - 1 \neq 0 \). That is, when \( x \neq \pm 1 \). Moreover,

\[
R'(x) = \frac{\left( \frac{d}{dx}(x + 2) \right) (x^2 - 1) - (x + 2) \left( \frac{d}{dx}(x^2 - 1) \right)}{(x^2 - 1)^2} = \frac{1 \cdot (x^2 - 1) - (x + 2)(2x)}{(x^2 - 1)^2} = \frac{(x^2 - 1) - 2x^2 - 4x}{(x^2 - 1)^2} = \frac{-x^2 - 4x - 1}{(x^2 - 1)^2}.
\]