Vertical Asymptotes and Infinite Limits

Created by

Barbara Forrest and Brian Forrest
Vertical Asymptotes

**Definition: [Right-Hand Infinite Limit]**

We say that the limit of \( f(x) \) as \( x \) approaches \( a \) from above (or from the right) is \( \infty \), if for every \( M > 0 \) there exists a \( \delta > 0 \) such that if \( a < x < a + \delta \), then

\[
\text{We write } \lim_{x \to a^+} f(x) = \infty.
\]
Vertical Asymptotes

**Note:** Similarly, we can define the right-hand infinite limit

\[
\lim_{x \to a^+} f(x) = -\infty
\]

and the two left-hand infinite limits

\[
\lim_{x \to a^-} f(x) = \pm \infty.
\]

**Definition: [Vertical Asymptote]**

We say that \( x = a \) is a vertical asymptote for \( f(x) \) if one of

\[
\lim_{x \to a^\pm} f(x) = \pm \infty
\]

holds.
**Important Note:** It is important to recognize that the symbol “±∞” is not a real number. When we say that “the limit of a function is ± infinity,” we are not saying that the limit exists in the proper sense. Instead, this expression simply provides useful information about the behavior of functions whose values become arbitrarily large, either positive or negative.

When we write expressions such as

$$\lim_{x \to a} f(x) = \infty,$$

we do **not** mean to imply that the function $f(x)$ has a limit at the point $x = a$. Instead, this expression actually tells us that the limit of the function **does not exist** precisely because the function **grows without bounds** near $a$. A similar statement can be made for all of the other cases. This is a subtle point but one of which you must be aware.
**Example:** The function \( f(x) = \frac{1}{x} \) has a vertical asymptote at \( x = 0 \) since

\[
\lim_{{x \to 0^-}} \frac{1}{x} = -\infty \quad \text{and} \quad \lim_{{x \to 0^+}} \frac{1}{x} = \infty.
\]
Fact: Let

\[ f(x) = \frac{p(x)}{q(x)} \]

be a rational function. Then \( f(x) \) has a vertical asymptote at \( x = a \) if and only if \( \lim_{x \to a} f(x) \) does not exist.

In particular, if

\[ q(a) = 0 \quad \text{and} \quad p(a) \neq 0, \]

then \( f(x) \) has a vertical asymptote at \( x = a \).
Example:

The function \( f(x) = \frac{x}{x^2 - 1} = \frac{x}{(x-1)(x+1)} \) has vertical asymptotes at \( x = -1 \) and \( x = 1 \).
**Example:** Consider

\[ f(x) = \frac{x^2 - 2x}{x^2 - x} = \frac{x(x - 2)}{x(x - 1)}. \]

Then \( q(x) = x^2 - x = 0 \) if \( x = 0 \) or \( x = 1 \), but only \( x = 1 \) is a vertical asymptote since \( \lim_{x \to 0} f(x) \) exists.
**Vertical Asymptotes and Rational Functions**

**Example:** Show that

\[
\lim_{x \to -2^+} \frac{x^2 - 1}{x + 2} = \infty \quad \text{and} \quad \lim_{x \to -2^-} \frac{x^2 - 1}{x + 2} = -\infty.
\]

Analyze the signs of \(x^2 - 1\) and \(x + 2\).