Continuity of Polynomials, Trigonometric Functions and Exponentials

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Recall: We showed that if

\[ p(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n, \]

then \( \lim_{x \to a} p(x) = p(a) \).

**Theorem: [Continuity of Polynomials]**

Let

\[ p(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n. \]

Then \( p(x) \) is continuous at each \( a \in \mathbb{R} \) with \( a > 0 \).
Continuity of $\sin(x)$ and $\cos(x)$

**Recall:** We have seen that

$$\lim_{x \to 0} \sin(x) = 0 = \sin(0) \quad \text{and} \quad \lim_{x \to 0} \cos(x) = 1 = \cos(0).$$

That is, both $\sin(x)$ and $\cos(x)$ are continuous at $a = 0$.

**Theorem:** [Continuity of $\sin(x)$ and $\cos(x)$]

Both $\sin(x)$ and $\cos(x)$ are continuous at each $a \in \mathbb{R}$.

**Proof:** Observe that

$$\lim_{x \to a} \sin(x) = \lim_{h \to 0} \sin(a + h) = \lim_{h \to 0} \sin(a) \cos(h) + \sin(h) \cos(a) = \sin(a) \cdot 1 + 0 \cdot \cos(a) = \sin(a).$$
Continuity of $\sin(x)$ and $\cos(x)$

Proof (continued): Next observe that

\[
\lim_{x \to a} \cos(x) = \lim_{h \to 0} \cos(a + h) = \lim_{h \to 0} \cos(a) \cos(h) - \sin(a) \sin(h) = \cos(a) \cdot 1 - \sin(a) \cdot 0 = \cos(a)
\]
Continuity of $e^x$

Note:

1) It is actually not an easy task to prove the continuity of the function $f(x) = e^x$.

2) The easiest way to show that $e^x$ is continuous is to realize that it can be defined by a special type of series known as a power series.

3) If $e^x$ is continuous at $x = 0$, then it is continuous everywhere.

First observe that if $e^x$ is continuous at $x = 0$, then

$$1 = e^0 = \lim_{h \to 0} e^h.$$  

Hence

$$\lim_{x \to a} e^x = \lim_{h \to 0} e^{a+h} = \lim_{h \to 0} e^a e^h = \lim_{h \to 0} e^a \lim_{h \to 0} e^h = e^a.$$
Theorem: [Continuity of Inverses]

Assume that $f(x)$ is invertible with inverse $g(y)$. If $f(a) = b$ and if $f(x)$ is continuous at $x = a$, then $g(y)$ is continuous at $y = b = f(a)$. 
Continuity of $e^x$ and $\ln(x)$

**Theorem:** [Continuity of $e^x$ and $\ln(x)$]

We have

1) $e^x$ is continuous at each $a \in \mathbb{R}$.

2) $\ln(x)$ is continuous for each $a \in \mathbb{R}$ with $a > 0$. 