One-sided Limits

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Recall: Formal Definition of a Limit
We say that $L$ is the limit of $f(x)$ as $x$ approaches $a$ if for every $\epsilon > 0$ there exists a $\delta > 0$ such that if

$$0 < |x - a| < \delta,$$

then

$$|f(x) - L| < \epsilon.$$
Example: Let \( f(x) = \frac{|x|}{x} \). \( \lim_{x \to 0} f(x) \) does not exist.
Example: Let $f(x) = \frac{|x|}{x}$. ⇒ $\lim_{x \to 0} f(x)$ does not exist.

If $x > 0$ and $x \to 0$, then $f(x) \to 1$. 
Example: Let $f(x) = \frac{|x|}{x}$. $\Rightarrow$ $\lim_{x \to 0} f(x)$ does not exist.

If $x > 0$ and $x \to 0$, then $f(x) \to 1$.

If $x < 0$ and $x \to 0$, then $f(x) \to -1$. 
One-sided Limits

Definition: [Limit from Above]

We say that $L$ is the limit of $f(x)$ as $x$ approaches $a$ from above (or from the right), if for every $\epsilon > 0$ there exists a $\delta > 0$ such that if $a < x < a + \delta$, then

$$|f(x) - L| < \epsilon.$$ 

We write $\lim_{x \to a^+} f(x) = L$. 
Definition: [Limit from Below]

We say that $L$ is the limit of $f(x)$ as $x$ approaches $a$ from below (or from the left), if for every $\epsilon > 0$ there exists a $\delta > 0$ such that if $a - \delta < x < a$, then

$$|f(x) - L| < \epsilon.$$ 

We write $\lim_{x \to a^-} f(x) = L$. 
Example: We know that \( \lim_{x \to 0} \frac{|x|}{x} \) does not exist. However,

\[
\lim_{x \to 0^+} \frac{|x|}{x} = 1
\]

and

\[
\lim_{x \to 0^-} \frac{|x|}{x} = -1.
\]
One-sided Limits vs. Limits

Theorem

The following are equivalent:

1. \( \lim_{x \to a} f(x) = L. \)

2. Both \( \lim_{x \to a^+} f(x) \) and \( \lim_{x \to a^-} f(x) \) exist with

\[
\lim_{x \to a^+} f(x) = L = \lim_{x \to a^-} f(x).
\]
One-sided Limits

Assume that \( \lim_{x \to a} f(x) = L \)

\[ \Rightarrow \lim_{x \to a^+} f(x) = L, \text{ and} \]
\[ \lim_{x \to a^-} f(x) = L. \]
One-sided Limits

Assume that \( \lim_{x \to a^+} f(x) = L = \lim_{x \to a^-} f(x) \).
One-sided Limits

Assume that 
\[ \lim_{x \to a^+} f(x) = L = \lim_{x \to a^-} f(x) \Rightarrow \lim_{x \to a} f(x) = L. \]