Limits of Functions: Part II

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Definition: [Limit of a Function]

We say that \( L \) is the limit of \( f(x) \) as \( x \) approaches \( a \) if for every \( \epsilon > 0 \) there exists a \( \delta > 0 \) such that if

\[
0 < |x - a| < \delta,
\]

then

\[
|f(x) - L| < \epsilon.
\]

We write

\[
\lim_{x \to a} f(x) = L.
\]
Proving the Limit Exists

Example: Show that
$$\lim_{x \to 2} 3x + 1 = 7.$$  
\[
\frac{\epsilon}{\delta} = \frac{\text{Rise}}{\text{Run}} = 3 \quad \Rightarrow \quad \delta = \frac{\epsilon}{3}.
\]

Algebraically, we want
\[
| (3x + 1) - 7 | < \epsilon \quad \iff \quad | 3x - 6 | < \epsilon \\
\quad \iff \quad 3 | x - 2 | < \epsilon \\
\quad \iff \quad | x - 2 | < \frac{\epsilon}{3}
\]

So if \( \delta = \frac{\epsilon}{3} \) and \( 0 < | x - 2 | < \delta \), we have \( | (3x + 1) - 7 | < \epsilon \).
**Remark:** If \( f(x) = mx + b \) where \( m \neq 0 \), then

\[
\lim_{x \to a} mx + b = m(a) + b.
\]

Given \( \epsilon > 0 \), if

\[
\delta = \frac{\epsilon}{|m|}
\]

and if \( 0 < |x - a| < \delta \), then

\[
| f(x) - (m(a) + b) | = |(mx + b) - (ma + b)|
= |m| \cdot |x - a|
< |m| \cdot \frac{\epsilon}{|m|}
= \epsilon
\]

**Example:**

\[
\lim_{x \to 1} \frac{x^2 - 1}{x - 1} = \lim_{x \to 1} x + 1 = 2.
\]
Example: Show that
\[ \lim_{x \to 3} x^2 = 9. \]

Let \( \epsilon > 0 \). We want
\[ 0 < |x - 3| < \delta \]
to imply
\[ |x^2 - 9| = |x - 3||x + 3| < \epsilon. \]

We might choose \( \delta = \frac{\epsilon}{|x+3|} \) since if \( 0 < |x - 3| < \frac{\epsilon}{|x+3|} \), then
\[ |x^2 - 9| < \frac{\epsilon}{|x + 3|} \cdot |x + 3| = \epsilon. \]

Note: \( \frac{\epsilon}{|x+3|} \) is not a constant!
**An Important Observation**

**Observation:** In the definition of a limit, if we find a $\delta$ that works for a particular $\epsilon$, then any smaller $\delta$ will also satisfy the definition of the limit of a function for the same $\epsilon$.

**Trick:** In showing that $\lim_{x \to 3} x^2 = 9$, we can always assume that $\delta \leq 1$. 
Example (continued)

If

$$0 < |x - 3| < \delta \leq 1,$$

then

$$2 < x < 4$$

so

$$|x + 3| < |4 + 3| = 7.$$ 

If $\delta < \min(1, \frac{\epsilon}{7})$, then

$$0 < |x - 3| < \delta \Rightarrow$$

$$|x^2 - 9| = |x - 3||x + 3| < \delta \cdot 7 < \frac{\epsilon}{7} \cdot 7 = \epsilon.$$