Limits of Functions: Part I

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What is a Limit?

Heuristic Definition:
We say that \( L \) is the limit of a function \( f(x) \) as \( x \) approaches \( a \) if as \( x \) gets closer and closer to \( a \), without ever reaching \( a \), \( f(x) \) gets closer and closer to \( L \).
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**Example:** Consider the two functions \( f(x) = \frac{x^2 - 1}{x - 1} \) and \( g(x) = x + 1 \). It might be tempting to use some algebra and write

\[
\begin{align*}
  f(x) &= \frac{x^2 - 1}{x - 1} \\
  &= \frac{(x + 1)(x - 1)}{x - 1} \\
  &= x + 1 \\
  &= g(x)
\end{align*}
\]

**Question:** Does this mean that \( f(x) \) and \( g(x) \) are actually the same function?

**Answer:** *Almost*, but not quite. They have different domains since \( f(x) \) is not defined at \( x = 1 \)!
What is a Limit?

Note: The graph of $g(x) = x + 1$ is a straight line with slope 1.

Question: What happens if we graph $f(x) = \frac{x^2 - 1}{x - 1}$?

Answer: The graph of $f(x)$ is the same graph as $g(x) = x + 1$, except there is a hole in the graph corresponding to where $x = 1$. 
What is a Limit?

We want to focus on the values of $f(x) = \frac{x^2 - 1}{x - 1}$ when $x$ is very close to but not equal to 1. The following is a table of some select values with $x$ near 1.

<table>
<thead>
<tr>
<th>$x$</th>
<th>$f(x)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>0.1</td>
<td>1.1</td>
</tr>
<tr>
<td>0.5</td>
<td>1.5</td>
</tr>
<tr>
<td>0.75</td>
<td>1.75</td>
</tr>
<tr>
<td>0.9</td>
<td>1.9</td>
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<tr>
<td>0.99</td>
<td>1.99</td>
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<tr>
<td>0.999</td>
<td>1.999</td>
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<tr>
<td>0.9999</td>
<td>1.9999</td>
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<tr>
<td>0.99999</td>
<td>1.99999</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$x$</th>
<th>$f(x)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>3</td>
</tr>
<tr>
<td>1.9</td>
<td>2.9</td>
</tr>
<tr>
<td>1.5</td>
<td>2.5</td>
</tr>
<tr>
<td>1.25</td>
<td>2.25</td>
</tr>
<tr>
<td>1.1</td>
<td>2.1</td>
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<tr>
<td>1.01</td>
<td>2.01</td>
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<tr>
<td>1.001</td>
<td>2.001</td>
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<td>2.00001</td>
</tr>
<tr>
<td>1.0000001</td>
<td>2.0000001</td>
</tr>
</tbody>
</table>
What is a Limit?

We can see that as $x$ gets closer and closer to 1, $f(x)$ gets closer and closer to 2.

We would like to say that 2 is the limit of $f(x)$ as $x$ approaches 1.
Formal Definition of a Limit

A more robust definition is required.

**Improved Heuristic Definition:**
$L$ is the limit of $f(x)$ as $x$ approaches $a$ if for any positive tolerance $\epsilon > 0$, we can ensure that $f(x)$ approximates $L$ with error less than $\epsilon$ at any $x$, other than possibly at $a$ itself, provided that $x$ is close enough to $a$.

**Definition: [Limit of a Function]**

We say that $L$ is the limit of $f(x)$ as $x$ approaches $a$ if for every $\epsilon > 0$ there exists a $\delta > 0$ such that if

$$0 < |x - a| < \delta,$$

then

$$|f(x) - L| < \epsilon.$$

We write

$$\lim_{x \to a} f(x) = L.$$
Formal Definition of a Limit

\[ |f(x) - L| < \epsilon \]

\[ |x - a| < \delta \]
Formal Definition of a Limit

\[ L - \epsilon < f(x) < L + \epsilon \]
\[ L - \epsilon_1 < f(x) < L + \epsilon_1 \]

\[ a - \delta < x < a + \delta \]
\[ a - \delta_1 < x < a + \delta_1 \]
Formal Definition of a Limit

Remarks:

1. For \( \lim_{x \to a} f(x) \) to exist, \( f(x) \) must be defined on an open interval \((\alpha, \beta)\) containing \( x = a \), except possibly at \( x = a \).

2. The value of \( f(a) \), if it is defined at all, does not affect the existence of the limit or its value.

3. If two functions are equal, except possibly at \( x = a \), then their limiting behavior at \( a \) is the same.