The Intermediate Value Theorem

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Equator Problem

Equator Problem:

Show that at any given time there will always be two diametrically opposite points on the equator with exactly the same temperature.
**Equator Problem (continued)**

**Strategy:**

1) Assume the equator is a circle and that each point can be identified by an angle $\theta$ in standard position with $T(\theta)$ the temperature at the point $\theta$.

2) Let

$$H(\theta) = T(\theta + \pi) - T(\theta).$$

3) Find a point $\theta_0$ with $H(\theta_0) = 0$. 
Central Problem: Solve

1. \( f(x) = 0 \)
2. \( f(x) = \alpha \)
3. \( f(x) = g(x) \)

Question: Does a solution exist? If so, how do you find it?
Central Problem

Observation:

1) Assume $f(x)$ is continuous on $[a, b]$ with $f(a) < \alpha < f(b)$.

2) To get from below the line $y = \alpha$ to above the line $y = \alpha$ without creating a break, we must cross the line $y = \alpha$ at least once so there exists $c \in (a, b)$ with $f(c) = \alpha$. 
Intermediate Value Theorem (IVT)

**Theorem: [The Intermediate Value Theorem (IVT)]**

Assume that $f(x)$ is continuous on the closed interval $[a, b]$, and either

$$f(a) < \alpha < f(b) \quad \text{or} \quad f(a) > \alpha > f(b).$$

Then there exists a $c \in (a, b)$ such that $f(c) = \alpha$. 

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**Graphical Representation:**

- $(a, f(a))$
- $(b, f(b))$
- $(c, f(c)) = (c, \alpha)$
- $y = \alpha$
- $a$ to $b$ interval
- $c$ point of intersection
Intermediate Value Theorem (IVT)

Note: The IVT appears to be very obvious but the proof that it is true is actually quite complicated and is beyond the scope of this course.
Recall:
**Equator Problem:**
Show that at any given time there will always be two diametrically opposite points on the equator with exactly the same temperature.

**Solution:**
Observe $H(\theta) = T(\theta + \pi) - T(\theta)$ is continuous on $[0, \pi]$. We have

\[
H(\pi) = T(\pi + \pi) - T(\pi) = T(2\pi) - T(\pi) = T(0) - T(\pi) = -H(0).
\]
Equator Problem

Given: $T(\pi + \pi) = T(2\pi) = T(0)$

Three Cases:

1) If $H(0) = 0$ we are done.

2) If $H(0) < 0$, then $H(\pi) > 0$ so the IVT gives $\theta_0$ with $H(\theta_0) = 0$.

3) If $H(0) > 0$, then $H(\pi) < 0$ so the IVT gives $\theta_0$ with $H(\theta_0) = 0$. 
Example:
Show that there exists a $c \in (0, 1)$ such that

$$
\cos(c) = c,
$$

or equivalently that there exists a $c \in (0, 1)$ with $h(c) = 0$ where

$$
h(x) = \cos(x) - x.
$$
Example (continued)

Solution: \( h(x) = \cos(x) - x \) is continuous on the closed interval \([0, 1]\) with

\[
\begin{align*}
  h(0) &= \cos(0) - 0 = 1 > 0 \\
  h(1) &= \cos(1) - 1 < 0.
\end{align*}
\]

By the IVT, we can conclude that there is a \( 0 < c < 1 \) such that 
\( h(c) = 0 \) or equivalently that \( \cos(c) = c \).