Definition: [Continuity]

We say that $f(x)$ is continuous at $x = a$ if

1) $\lim_{x \to a} f(x)$ exists.

2) $\lim_{x \to a} f(x) = f(a)$.

Otherwise, we say that $f(x)$ is discontinuous at $a$ or that $a$ is a point of discontinuity for $f(x)$. 
Definition of Continuity

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Note: 2) $\Rightarrow$ 1).
Definition of Continuity

**Observe:** The following are equivalent:

1) \( \lim_{x \to a} f(x) = f(a) \).

2) For every \( \epsilon > 0 \), there exists a \( \delta > 0 \) such that
   \[ 0 < |x - a| < \delta \implies |f(x) - f(a)| < \epsilon. \]

3) For every \( \epsilon > 0 \), there exists a \( \delta > 0 \) such that
   \[ |x - a| < \delta \implies |f(x) - f(a)| < \epsilon. \]

**Alternate Definition:** [Continuity]

We say that \( f(x) \) is continuous at \( x = a \) if for every \( \epsilon > 0 \), there exists a \( \delta > 0 \) such that if \( |x - a| < \delta \), then

\[ |f(x) - f(a)| < \epsilon. \]
Definition of Continuity

\[ f(a) = \lim_{x \to a} f(x) \]

\[ f(a) - \epsilon < f(x) < f(a) + \epsilon \]
Definition of Continuity

\[ f(a) + \epsilon \]
\[ f(a) - \epsilon \]
\[ f(a) + \epsilon_1 \]
\[ f(a) - \epsilon_1 \]
Sequential Characterization of Continuity

**Recall:** The following are equivalent:

1) \( \lim_{x \to a} f(x) = L \).

2) If \( \{x_n\} \) is a sequence with \( x_n \to a \) and \( x_n \neq a \), then \( \lim_{n \to \infty} f(x_n) = L \).

**Theorem: [Sequential Characterization of Continuity]**

The following are equivalent:

1) \( f(x) \) is continuous at \( x = a \).

2) If \( \{x_n\} \) is a sequence with \( x_n \to a \), then \( \lim_{n \to \infty} f(x_n) = f(a) \).