Continuity on an Interval

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Continuity on an Open Interval

Definition: [Continuity on an Open Interval]

We say that \( f(x) \) is continuous on the open interval \( I \) if it is continuous at every \( a \in I \).

Problem: How does this work for closed intervals?

Is \( f(x) = \sqrt{x} \) continuous at 0?

Answer: \( \lim_{x \to 0} \sqrt{x} \) does not exist.
One-sided Continuity

Definition: [One-sided Continuity]

We say that $f(x)$ is continuous from the left at $x = a$ if

$$\lim_{x \to a^-} f(x) = f(a).$$

We say that $f(x)$ is continuous from the right at $x = a$ if

$$\lim_{x \to a^+} f(x) = f(a).$$

Example: $f(x) = \sqrt{x}$ is continuous from the right at 0.
Continuity on a Closed Interval

**Definition: [Continuity on a Closed interval]**

We say that \( f(x) \) is continuous on the closed interval \([a, b]\) if:

1) It is continuous on \((a, b)\). That is \( f(x) \) is continuous at every \( x \in (a, b) \) in the usual sense.

2) \( \lim_{{x \to a^+}} f(x) = f(a) \).

3) \( \lim_{{x \to b^-}} f(x) = f(b) \).
Example: Let $f(x) = \sqrt{1 - x^2}$.

We have that

$$f(-1) = \lim_{x \to -1^+} \sqrt{1 - x^2} = 0 = \lim_{x \to 1^-} \sqrt{1 - x^2} = f(1)$$

so $f(x)$ is continuous on $[-1, 1]$. 
Continuity on an Interval

**Observation:** The following are equivalent:

1) $f(x)$ is continuous on an interval $I$.

2) If \ \{x_n\} is a sequence in $I$ with $x_n \to x_0 \in I$, then
   $$f(x_n) \to f(x_0).$$