Limits of Sequences

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Definition of a Limit of a Sequence

Recall:

Formal Definition: [Limit of a Sequence]

We say that $L$ is the limit of the sequence $\{a_n\}$ as $n$ goes to infinity if for every $\epsilon > 0$, there exists an $N \in \mathbb{N}$ such that if $n \geq N$, then

$$|a_n - L| < \epsilon.$$ 

In this case, we write

$$\lim_{n \to \infty} a_n = L.$$ 

We may also say $\{a_n\}$ converges to $L$ and write $a_n \to L$.

If no such $L$ exists, we say that $\{a_n\}$ diverges.
Example 1: Show that \( \lim_{n \to \infty} \frac{1}{\sqrt{n}} = 0 \).

Let \( \epsilon > 0 \). We need to find a cutoff \( N \) that satisfies the definition of the limit.

- If \( \frac{1}{\epsilon^2} < n \Rightarrow \frac{1}{n} < \epsilon^2 \Rightarrow \frac{1}{\sqrt{n}} < \epsilon \).

- Hence, if \( \frac{1}{\epsilon^2} < N \), then \( n \geq N \Rightarrow | \frac{1}{\sqrt{n}} - 0 | < \epsilon \).

Therefore, we have shown the limit is 0.
Example 2: It can be shown that
\[
\lim_{{n \to \infty}} \frac{3n + 2}{4n + 3} = \frac{3}{4}.
\]
Find a cutoff \( N \) so that if \( n \geq N \), then
\[
\left| \frac{3n + 2}{4n + 3} - \frac{3}{4} \right| < \frac{1}{1000}.
\]
Solution: Observe that
\[
\left| \frac{3n + 2}{4n + 3} - \frac{3}{4} \right| = \left| \frac{12n + 8}{16n + 12} - \frac{12n + 9}{16n + 12} \right| = \frac{-1}{16n + 12} = \frac{1}{16n + 12}
\]
We want
\[
\frac{1}{16n + 12} < \frac{1}{1000} \Rightarrow 1000 < 16n + 12 \Rightarrow 61.75 < n, \text{ so } N = 62.
\]
Example 3:
Consider \( \{(−1)^{n+1}\} = \{1, -1, 1, -1, \ldots\} \).
Does \( \{(−1)^{n+1}\} \) have a limit?
Is \( \lim_{n \to \infty} \{(−1)^{n+1}\} = 1? \)
Is \( \lim_{n \to \infty} \{(−1)^{n+1}\} = -1? \)
Or both?

Solution: Assume \( \lim_{n \to \infty} \{(−1)^{n+1}\} = L \) and \( \epsilon = \frac{1}{2} \). Choose the
cutoff \( N \) such that if \( n \geq N \) then,

\[
| (−1)^{n+1} - L | < \frac{1}{2}.
\]
Pick \( k_1 \in \mathbb{N} \) such that \( 2k_1 \geq N \).

Then

\[
 a_{2k_1} = -1
\]

\[ \implies \quad | -1 - L | < \frac{1}{2} \]
Example 3 (continued)

Pick $k_1 \in \mathbb{N}$ such that $2k_1 \geq N$.

Then

$$a_{2k_1} = -1$$

$$\Rightarrow | -1 - L | < \frac{1}{2}$$

$$\Rightarrow L \in \left( -\frac{3}{2}, -\frac{1}{2} \right)$$

Hence, $L \in \left( -\frac{3}{2}, -\frac{1}{2} \right)$ and $L \in \left( \frac{1}{2}, \frac{3}{2} \right)$ which is impossible.

Therefore, $\{(-1)^{n+1}\}$ has no limit!
**Uniqueness of Limits**

**Problem:** Can \(\{a_n\}\) have two different limits?

Assume \(\lim_{n \to \infty} a_n = L\) and \(\lim_{n \to \infty} a_n = M\) with \(L < M\).

Consider \(\frac{M+L}{2}\). Let \(\epsilon = \frac{M-L}{2}\).

Consider \(a_{n_0}\). If \(n_0\) is large enough, then

\[ a_{n_0} \in (M - \epsilon, M + \epsilon) \]

and

\[ a_{n_0} \in (L - \epsilon, L + \epsilon) \]

which is impossible!
Theorem: [Uniqueness of Limits]

Assume that \( \lim_{n \to \infty} a_n = L \) and \( \lim_{n \to \infty} a_n = M \). Then

\[ L = M. \]
Note: It is often difficult to tell if a sequence converges and if so, what its limit might be.

Example 4: Consider the recursively defined sequence

\[ a_1 = 1, \quad a_{n+1} = \cos(a_n). \]

Does \( \{a_n\} \) converge? If so, what is \( \lim_{n \to \infty} a_n \)?
Example 4 (continued)

\[ y = x \]

\[ a_1 = 1, \]

\[ \cos(x) \]
Example 4 (continued)

\[ y = x \]

\[ a_1 = 1, \]
\[ a_2 = 0.5403023059, \]
Example 4 (continued)

\[ y = x \]

\[ a_1 = 1, \]
\[ a_2 = 0.5403023059, \]
Example 4 (continued)

\[ y = x \]

\[ a_1 = 1, \]
\[ a_2 = 0.5403023059, \]

\[ \cos(x) \]
Example 4 (continued)

\[ y = x \]

\[ a_1 = 1, \]
\[ a_2 = 0.5403023059, \]
\[ a_3 = 0.8575532158, \]
Example 4 (continued)

\[ y = x \]

\[ \cos(x) \]

\[ a_1 = 1, \]
\[ a_2 = 0.5403023059, \]
\[ a_3 = 0.8575532158, \]
\[ a_4 = 0.6542897905, \]
Example 4 (continued)

$\cos(x) = x$

$a_1 = 1,$
$a_2 = 0.5403023059,$
$a_3 = 0.8575532158,$
$a_4 = 0.6542897905,$
$a_5 = 0.7934803587,$
Example 4 (continued)

\[ y = x \]

\[ \cos(x) \]

\[ a_1 = 1, \]
\[ a_2 = 0.5403023059, \]
\[ a_3 = 0.8575532158, \]
\[ a_4 = 0.6542897905, \]
\[ a_5 = 0.7934803587, \]
\[ a_6 = 0.7013687737, \]
Example 4 (continued)

\[ y = x \]

\[ \cos(x) \]

\[ a_1 = 1, \]
\[ a_2 = 0.5403023059, \]
\[ a_3 = 0.8575532158, \]
\[ a_4 = 0.6542897905, \]
\[ a_5 = 0.7934803587, \]
\[ a_6 = 0.7013687737, \]
\[ a_7 = 0.7639596829, \]
\[ a_8 = 0.7221024250, \]
\[ a_9 = 0.7504177618, \]
Example 4 (continued)

\[ y = x \]

\[ \cos(x) \]

\[ a_1 = 1, \]
\[ a_2 = 0.5403023059, \]
\[ a_3 = 0.8575532158, \]
\[ a_4 = 0.6542897905, \]
\[ a_5 = 0.7934803587, \]
\[ a_6 = 0.7013687737, \]
\[ a_7 = 0.7639596829, \]
\[ a_8 = 0.7221024250, \]
\[ a_9 = 0.7504177618, \]

0.7314040424, 0.7442373549, 0.7356047404, 0.7414250866, 0.7375068905, 0.7401473356, 0.7383692041, 0.7395672022, 0.7387603199, 0.7393038924, 0.7389377567, 0.7391843998, ...
Example 4 (continued)

\[ y = x \]

\[ \cos(x) = x \]

\[ a_{72} = 0.7390851332, \quad a_{73} = 0.7390851332, \quad \text{and} \quad a_{74} = 0.7390851332 \]

suggest that \( \{a_n\} \) converges to some \( L \).
Example 4 (continued)

\[ y = x \]

\[ \cos(x) \]

\[ a_{72} = 0.7390851332, \]
\[ a_{73} = 0.7390851332, \]

and

\[ a_{74} = 0.7390851332 \]

suggest that \( \{a_n\} \) converges to some \( L \).

In fact,

\[ \cos(L) = L. \]