Limits of Sequences

Created by

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The Limit of a Sequence

New Heuristic Definition:
We say that $L$ is the limit of the sequence $\{a_n\}$ as $n$ goes to infinity if no matter what positive tolerance $\epsilon > 0$ we are given, we can find a cutoff $N \in \mathbb{N}$ such that the terms $a_n$ approximate $L$ with an error less than $\epsilon$ provided that $n \geq N$.

Formal Definition: [Limit of a Sequence]

We say that $L$ is the limit of the sequence $\{a_n\}$ as $n$ goes to infinity if for every $\epsilon > 0$, there exists an $N \in \mathbb{N}$ such that if $n \geq N$, then

$$|a_n - L| < \epsilon.$$ 

In this case, we say that $\{a_n\}$ converges to $L$ and we write

$$\lim_{n \to \infty} a_n = L.$$ 

If no such $L$ exists we say that $\{a_n\}$ diverges.
1. Identify $L$.
2. Specify the error $\epsilon > 0$.
3. Find the cutoff $N$.
4. Choose a smaller $\epsilon_1$.
5. **Repeat Step 3** with a larger $N_1$. 

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- Not all terms in \( \{a_n\} \) must fall in \((L - \epsilon, L + \epsilon)\).
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If \(n\) is large enough, then \(a_n \in (L - \epsilon, L + \epsilon)\).
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Then

\[ (L - \epsilon, L + \epsilon) \subseteq (a, b) \.

If \( n \) is large enough, then \( a_n \in (L - \epsilon, L + \epsilon) \) and hence

\[ a_n \in (a, b) \.

**Theorem**

The following statements are equivalent:

1. \( \lim_{n \to \infty} a_n = L \).
2. Every interval \((L - \epsilon, L + \epsilon)\) contains a tail of \(\{a_n\}\).
3. Every interval \((L - \epsilon, L + \epsilon)\) contains all but finitely many terms of \(\{a_n\}\).
4. Every interval \((a, b)\) containing \(L\) contains a tail of \(\{a_n\}\).
5. Every interval \((a, b)\) containing \(L\) contains all but finitely many terms of \(\{a_n\}\).

**Important Note:** Changing finitely many terms in \(\{a_n\}\) does not affect convergence.