Absolute Value and the Triangle Inequality

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Absolute Value

**Definition: [Absolute Value]**

The *absolute value* of a real number $x$ is the quantity

$$|x| = \begin{cases} x & \text{if } x \geq 0, \\ -x & \text{if } x < 0. \end{cases}$$

**Properties:**

1) $|x| \geq 0$

2) $|x| = |-x|$
Geometric Interpretation of Absolute Value

Remark: \(|x| = \sqrt{x^2}\)

The absolute value is the one-dimensional analogue of \(\sqrt{x^2 + y^2}\), which measures the “length” of a vector \((x, y)\) in the plane.

Geometric Interpretation:

- \(|x| = \) the distance from \(x\) to 0.
- \(|x - a| = \) the distance from \(x\) to \(a\).
**Triangle Inequality**

**Theorem: [Triangle Inequality]**

For any $x, y, z \in \mathbb{R}$

$$|x - y| \leq |x - z| + |z - y|$$

**Remark:** The length of any side of a triangle is less than or equal to the sum of the other two sides.
Proof of the Triangle Inequality

Proof:

We may assume $x < y$. There are three cases:

Case 1: $z < x$

Then $|x - y| < |z - y| \leq |x - z| + |z - y|$
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Case 2: $x \leq z \leq y$
Proof of the Triangle Inequality

Case 2: \( x \leq z \leq y \)

Then

\[ |x - y| \]
Proof of the Triangle Inequality

Case 2: $x \leq z \leq y$

Then

$$|x - y| = |x - z|$$
Proof of the Triangle Inequality

Case 2: \( x \leq z \leq y \)

Then

\[ |x - y| = |x - z| + |z - y| \]
Proof of the Triangle Inequality

Case 3: $y < z$
Proof of the Triangle Inequality

Case 3: $y < z$

Then

$$|x - y|$$
Proof of the Triangle Inequality

Case 3: $y < z$

Then

$$|x - y| < |x - z|$$
Proof of the Triangle Inequality

Case 3: $y < z$

Then

$$|x - y| < |x - z| \leq |x - z| + |z - y|$$
Theorem: [Triangle Inequality II]

Let $x, y \in \mathbb{R}$. Then

$$|x + y| \leq |x| + |y|$$

Proof: Let $x, y \in \mathbb{R}$. Applying the Triangle Inequality to $x, -y$ and $z = 0$ gives

$$|x + y| = |x - (-y)|$$

$$\leq |x - 0| + |0 - (-y)|$$

$$= |x| + |y|$$
Example

Problem: Find all $x \in \mathbb{R}$ such that $|x - 3| < 2$.

Approach 1: Algebraic Solution

$$|x - 3| < 2 \iff -2 < x - 3 < 2 \iff -2 + 3 < x < 2 + 3$$

Solution: $x \in (1, 5)$. 
Example

\[ |x - 3| < 2 \]

Approach 2: Geometric Solution
Example

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“distance from $x$ to 3 is less than 2”
Example

| x − 3 | < 2

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"distance from $x$ to 3 is less than 2" $\implies x > 1$
Example

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“distance from $x$ to 3 is less than 2” $\implies x > 1$ and $x < 5$. 

$|x - 3| < 2$
Example

Approach 2: Geometric Solution

“distance from $x$ to 3 is less than 2” $\implies x > 1$ and $x < 5$.

Solution: $x \in (1, 5)$. 
Important Inequalities

1. \( |x - a| \leq \delta \) if and only if \( x \in [a - \delta, a + \delta] \).

2. \( |x - a| < \delta \) if and only if \( x \in (a - \delta, a + \delta) \).

3. \( 0 < |x - a| < \delta \) if and only if \( x \in (a - \delta, a + \delta) \) \( \{a\} \).
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1. $|x - a| \leq \delta$ if and only if $x \in [a - \delta, a + \delta]$.
2. $|x - a| < \delta$ if and only if $x \in (a - \delta, a + \delta)$.
3. $0 < |x - a| < \delta$ if and only if $x \in (a - \delta, a + \delta) \setminus \{a\}$.