Least Upper Bound Property

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Bounded Sets

Definition: [Upper/Lower Bounds]

Let $S \subset \mathbb{R}$.

1. $\alpha$ is an upper bound for $S$ if $x \leq \alpha$ for all $x \in S$. 
Definition: [Upper/Lower Bounds]

Let $S \subset \mathbb{R}$.

1. $\alpha$ is an upper bound for $S$ if $x \leq \alpha$ for all $x \in S$.
2. $\beta$ is a lower bound for $S$ if $\beta \leq x$ for all $x \in S$. 
Bounded Sets

Definition: [Bounded Sets]

Let \( S \subset \mathbb{R} \).

1. \( S \) is bounded above if \( S \) has an upper bound \( \alpha \).
2. \( S \) is bounded below if \( S \) has a lower bound \( \beta \).
Bounded Sets

Definition: [Bounded Sets]

Let $S \subseteq \mathbb{R}$.

1. $S$ is **bounded above** if $S$ has an upper bound $\alpha$.
2. $S$ is **bounded below** if $S$ has a lower bound $\beta$.
3. $S$ is **bounded** if $S$ is bound above and bounded below.
The Set \([0, 1)\)

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- \(S\) is bounded above by 2.
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- $S$ is also bounded above by 4.
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- $S$ is also bounded above by 4.
- $S$ has infinitely many upper bounds.
- 1 is a special upper bound for $S$.
- 1 is the smallest or least upper bound for $S$. 
The Set $[0, 1)$

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The Set $[0, 1)$
Let \( S = [0, 1) \).

- \( S \) is bounded below by \(-1\).
- \( S \) is also bounded below by \(-2\).
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- $S$ is bounded below by $-1$.
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- $S$ has infinitely many lower bounds.
Let $S = [0, 1)$.

- $S$ is bounded below by $-1$.
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- $0$ is a special lower bound for $S$. 

The Set $[0, 1)$
Let \( S = [0, 1) \).

- \( S \) is bounded below by \(-1\).
- \( S \) is also bounded below by \(-2\).
- \( S \) has infinitely many lower bounds.
- 0 is a special lower bound for \( S \).
- 0 is the largest or greatest lower bound for \( S \).
Definition: [Least Upper Bound]

We say that $\alpha \in \mathbb{R}$ is the least upper bound for a set $S \subset \mathbb{R}$ if

1. $\alpha$ is an upper bound for $S$, and
2. if $\gamma$ is an upper bound for $S$, then $\alpha \leq \gamma$.

If a set $S$ has a least upper bound, then we denote it by $lub(S)$.

The least upper bound of $S$ is often called the supremum of $S$, denoted by $\text{sup}(S)$. 
Definition: [Greatest Lower Bound]

We say that $\beta \in \mathbb{R}$ is the greatest lower bound for a set $S \subset \mathbb{R}$ if

1. $\beta$ is a lower bound for $S$, and
2. if $\gamma$ is a lower bound for $S$, then $\beta \geq \gamma$.

If a set $S$ has a greatest bound, then we denote it by $glb(S)$.

The greatest lower bound of $S$ is often called the infimum of $S$, denoted by $inf(S)$. 
The Set $[0, 1)$

Let $S = [0, 1)$. 
Let \( S = [0, 1) \).

\[ \text{lub}(S) = 1 \]

\[ \text{glb}(S) = 0 \]

Note: \( \text{glb}(S) \in S \), but \( \text{lub}(S) \not\in S \).
Let $S = [0, 1)$.

- $lub(S) = 1$
- $glb(S) = 0$
Let $S = [0, 1)$.

- $\text{lub}(S) = 1$
- $\text{glb}(S) = 0$

**Note:** $\text{glb}(S) = 0 \in S$,
The Set $[0, 1)$

Let $S = [0, 1)$.

- $\text{lub}(S) = 1$
- $\text{glb}(S) = 0$

**Note:** $\text{glb}(S) = 0 \in S$, but $\text{lub}(S) = 1 \notin S$. 
Maximum and Minimum

**Definition: [Maximum/Mininum]**

1. If \( S \) contains \( \alpha = lub(S) \), then \( \alpha \) is called the *maximum* of \( S \) and is denoted by \( \max(S) \).
2. If \( S \) contains \( \beta = glb(S) \), then \( \beta \) is called the *minimum* of \( S \) and is denoted by \( \min(S) \).

**Example:** If \( S \) is a finite set with \( n \) elements

\[
S = \{a_1 < a_2 < \cdots < a_n\},
\]

then

- \( a_n = lub(S) = \max(S) \), and
- \( a_1 = glb(S) = \min(S) \).
The Empty Set

**Problem:** Does every set $S$ that is bounded above have a LUB?

**Axiom:** [Least Upper Bound Property or LUBP]

A *nonempty* subset $S \subseteq \mathbb{R}$ that is bounded above always has a least upper bound.
Example

**Example:** Let $S$ be the terms in the sequence $\{1 - \frac{1}{n}\}$.

That is,

$$S = \{0, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \cdots \}.$$

**Note:** Each term is less than 1, but we can get as close to 1 as we would like so long as the index $n$ is large enough.

Hence,

$$1 = \text{lub}(S).$$

We also know that

$$1 = \lim_{n \to \infty} \left(1 - \frac{1}{n}\right).$$

The fact that the limit and the least upper bound agree is no accident.