**Geometric Series**

**Definition:** [Geometric Series]

Let \( r \in \mathbb{R} \). Then

\[
\sum_{n=0}^{\infty} r^n = 1 + r + r^2 + r^3 + \cdots
\]

is called a *geometric series* of radius \( r \).

**Problem:** For which \( r \) does the geometric series \( \sum_{n=0}^{\infty} r^n \) converge?

To answer this question, we must look at its sequence of partial sums:

\[
S_k = \sum_{n=0}^{k} r^n = 1 + r + r^2 + \cdots + r^k.
\]
Geometric Series

**Problem:** For which \( r \) does the series \( \sum_{n=0}^{\infty} r^n \) converge?

**Case 1:** \( r = 1 \)

\[
S_k = 1 + 1 + 1 + \cdots + 1 = k + 1.
\]

Since \( \{S_k\} = \{k + 1\} \) diverges, the series \( \sum_{n=0}^{\infty} 1^n \) diverges.
Geometric Series

**Problem:** For which $r$ does the series $\sum_{n=0}^{\infty} r^n$ converge?

**Case 2:** $r = -1$

\[
S_k = 1 + (-1) + 1 + \cdots + (-1)^k = \begin{cases} 
1 & \text{if } k \text{ is even,} \\
0 & \text{if } k \text{ is odd.}
\end{cases}
\]

Since $\{S_k\} = \{1, 0, 1, 0, 1, 0, \ldots\}$ diverges, $\sum_{n=0}^{\infty} (-1)^n$ diverges.
Geometric Series

**Problem:** For which \( r \) does the series \( \sum_{n=0}^{\infty} r^n \) converge?

**Case 3:** \( r \neq 1 \)

\[
S_k = 1 + r + r^2 + \cdots + r^k
\]

\[
rS_k = r + r^2 + \cdots + r^k + r^{k+1}
\]

\[
\Rightarrow (1 - r)S_k = 1 - r^{k+1}
\]

\[
\Rightarrow S_k = \frac{1 - r^{k+1}}{1 - r}
\]
**Problem:** For which $r$ does the series $\sum_{n=0}^{\infty} r^n$ converge?

Now

$$|r^{k+1}| \rightarrow \begin{cases} 0 & \text{if } |r| < 1, \\ \infty & \text{if } |r| > 1. \end{cases}$$

But if $S_k = \frac{1 - r^{k+1}}{1 - r}$,

$$\lim_{k \to \infty} S_k = \begin{cases} \frac{1}{1 - r} & \text{if } |r| < 1, \\ \text{does not exist} & \text{if } |r| \geq 1. \end{cases}$$
Geometric Series

Theorem: [Geometric Series Test]

A geometric series \( \sum_{n=0}^{\infty} r^n \) converges if and only if \(|r| < 1\).

Moreover, if \(|r| < 1\),

\[
\sum_{n=0}^{\infty} r^n = \frac{1}{1 - r}
\]

Example:

\[
\sum_{n=0}^{\infty} \frac{1}{2^n} = \frac{1}{1 - \frac{1}{2}} = 2.
\]