MATROIDS DENSER THAN A CLIQUE

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ABSTRACT. The *growth-rate function* for a minor-closed class $\mathcal{M}$ of matroids is the function h where, for each non-negative integer r, $h(r)$ is the maximum number of elements of a simple matroid in $\mathcal{M}$ with rank at most r. The Growth-Rate Theorem of Geelen, Kabell, Kung, and Whittle shows, essentially, that the growth-rate function is always either linear, quadratic, exponential, or infinite. Morover, if the growth-rate function is quadratic, then $h(r) \geq \binom{r+1}{2}$, with the lower bound coming from the fact that such classes necessarily contain all graphic matroids. We characterise the classes that satisfy $h(r) = \binom{r+1}{2}$ for all sufficiently large r.

1. INTRODUCTION

An *extension* of a matroid M by an element $e \notin E(M)$ is a matroid M' such that $M = M' \backslash e$. An extension of $M \cong M(K_{n+1})$ by e is nongraphic if and only if e is not a loop or a coloop or parallel to any other element of M. We prove the following theorem:

Theorem 1.1. *Let $n \geq 2$ and $\ell \geq 3$ be integers. If M is a simple matroid of sufficiently large rank with $|M| > \binom{r(M)+1}{2}$, then M has a minor isomorphic to either $U_{2,\ell+2}$ or a nongraphic extension of $M(K_{n+1})$.*

This theorem is closely related to the problem of determining growth rates of minor-closed classes. For a class $\mathcal{M}$ of matroids containing the empty matroid, let $h_{\mathcal{M}}(n) : \mathbb{Z}_0^+ \to \mathbb{Z}_0^+ \cup \{\infty\}$ denote the *growth rate function* of $\mathcal{M}$: the function whose value at an integer $n \geq 0$ is given by the maximum number of elements in a simple matroid in $\mathcal{M}$ of rank at most n. For example, the class $\mathcal{G}$ of graphic matroids has growth rate function $h_{\mathcal{G}}(n) = \binom{n+1}{2}$. Any class containing all simple rank-2 matroids has infinite growth rate function for all $n \geq 2$; the following theorem of Geelen, Kabell, Kung and Whittle (see [6]) determines all growth rate functions to within a constant factor. To simplify the

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statement of this, and other results, we will take the convention that *minor-closed* classes of matroids are closed under both minors and isomorphism.

Theorem 1.2 (Growth Rate Theorem). *If $\mathcal{M}$ is a nonempty minor-closed class of matroids not containing all simple rank-2 matroids, then there exists $c \in \mathbb{R}^+$ so that either:*

(1) $h_{\mathcal{M}}(n) \le cn$ for all n,
(2) $\binom{n+1}{2} \le h_{\mathcal{M}}(n) \le cn^2$ for all n and $\mathcal{M}$ contains all graphic matroids, or
(3) there is a prime power q such that $\frac{q^n-1}{q-1} \le h_{\mathcal{M}}(n) \le cq^n$ for all n and $\mathcal{M}$ contains all $\mathrm{GF}(q)$-representable matroids.

Minor-closed classes satisfying (2) are *quadratically dense.* If f and g are functions, then we write $f(n) \approx g(n)$ if $f(n) = g(n)$ for all but finitely many n. Theorem 1.1 will imply a stronger result, Theorem 1.5, which in turn implies the following theorem, giving a 'gap' in which no growth rate function can fall.

Theorem 1.3. *Let $\mathcal{M}$ be a quadratically dense minor-closed class of matroids. Either $h_{\mathcal{M}}(n) \approx \binom{n+1}{2}$, or $h_{\mathcal{M}}(n) \ge \binom{n+2}{2} - 3$ for all $n \ge 2$.*

Similar behaviour has been shown to occur in the 'exponentially dense' case; see Nelson [12, Theorem 1.5.13].

Exponentially dense classes are easier to work with than quadratically dense classes since the extremal matroids are very highly connected; see [12, Theorem 1.5.6]. In fact, the extremal matroids are "weakly round", implying that this connectivity is not lost by contraction. For quadratically dense classes, one can show that the extremal matroids are highly connected and that there are "useful minors", but it is not straightforward to find useful minors that are sufficiently connected. Perhaps the main contribution of this paper is a technical result, Theorem 5.1, that resolves this issue. We anticipate that this result will prove useful for determining growth rate functions of other quadratically dense classes: for example, the *golden mean matroids* representable over GF(4) and GF(5), which are conjectured by Archer [1] to have a growth rate function of $\binom{n+3}{2} - 5$ for all $n \ge 4$.

Unavoidable Minors. For each integer $n \ge 3$, let D_n denote the binary incidence matrix of K_n, and let $M_n^{\square}$ denote the matroid $M(D_n|v)$, where v is a binary column vector with exactly four nonzero entries. Let $M_n^{\triangle}$ denote the principal extension of a triangle in $M(K_n)$, and let M_n° denote the free extension of $M(K_n)$. We also prove the following theorem:

Theorem 1.4. *Let m, n be integers so that $m \geq 4$ and $n \geq 2m^2$. If M is a nongraphic extension of $M(K_n)$, then M has a minor isomorphic to $M_m^{\square}, M_m^{\triangle}$, or M_m°.*

This gives us a stronger version of Theorem 1.1. Let $\mathcal{G}^{\square}$ denote the closure under minors of the set $\{M_n^{\square} : n \geq 4\}$, and define $\mathcal{G}^{\triangle}$ and $\mathcal{G}^{\circ}$ similarly. One can routinely show the following characterisations of these three classes (see [16] for the definitions of even-cycle and signed-graphic matroids):

- $\mathcal{G}^{\square}$ is the class of even-cycle matroids represented by a signed graph with a blocking pair.
- $\mathcal{G}^{\triangle}$ is the class of signed-graphic matroids represented by a signed graph having a vertex incident with all negative non-loop edges.
- $\mathcal{G}^{\circ}$ is the union of the classes of graphic matroids and truncations of graphic matroids.

Theorem 1.1 combined with Theorem 1.4 gives the following:

Theorem 1.5. *Let $\mathcal{M}$ be a quadratically dense minor-closed class of matroids. Either*

(1) $h_{\mathcal{M}}(n) \approx \binom{n+1}{2}$, or
(2) $\mathcal{M}$ contains $\mathcal{G}^{\circ}, \mathcal{G}^{\triangle}$, or $\mathcal{G}^{\square}$.

We obtain Theorem 1.3 from the above by computing the growth rate functions of the three classes; these follow easily from our characterisations:

Lemma 1.6. *For all $n \geq 2$,*

- $h_{\mathcal{G}^{\square}}(n) = \binom{n+2}{2} - 3$,
- $h_{\mathcal{G}^{\triangle}}(n) = \binom{n+2}{2} - 2$, *and*
- $h_{\mathcal{G}^{\circ}}(n) = \binom{n+2}{2}$.

Finite Fields. The three classes in Theorem 1.5 are not all representable over all finite fields; this allows us to obtain stronger statements when every matroid in $\mathcal{M}$ is representable over some fixed finite field. For any such field $\mathbb{F}$, the co-line $U_{\ell,\ell+2}$ is not $\mathbb{F}$-representable but is the truncation of the circuit $U_{\ell+1,\ell+2}$ so is in $\mathcal{G}^{\circ}$. Therefore not every matroid in $\mathcal{G}^{\circ}$ is $\mathbb{F}$-representable.

Note that $U_{2,4} \cong M_3^{\triangle} \in \mathcal{G}^{\triangle}$. Thus, if $\mathcal{M}$ contains only binary matroids, then $\mathcal{G}^{\triangle} \not\subseteq \mathcal{M}$ and so $\mathcal{G}^{\square} \subseteq \mathcal{M}$ is the only outcome. By our characterisation of $\mathcal{G}^{\square}$, this gives the following:

Corollary 1.7. *If $\mathcal{M}$ is a minor-closed class of binary matroids, then $h_{\mathcal{M}}(n) \approx \binom{n+1}{2}$ if and only if $\mathcal{M}$ contains all graphic matroids but not all matroids in $\mathcal{G}^{\square}$.*

Nongraphic matroids in $\mathcal{G}^{\circ}$ include the Fano matroid F_7, the affine geometry $\mathrm{AG}(3,2)$, and its unique simple rank-4 binary extension. Therefore our theorem implies (for large n) growth-rate results for excluding these matroids proved respectively by Heller [8], Kung et al. [9] and McGuinness [11].

Note that $F_7 \cong M_4^{\square} \in \mathcal{G}^{\square}$, so if $\mathcal{M}$ contains only matroids representable over some finite field of odd order, then $\mathcal{G}^{\square} \not\subseteq \mathcal{M}$. Thus we have the following:

Corollary 1.8. *Let q be an odd prime power. If $\mathcal{M}$ is a minor-closed class of $\mathrm{GF}(q)$-representable matroids, then $h_{\mathcal{M}}(n) \approx \binom{n+1}{2}$ if and only if $\mathcal{M}$ contains all graphic matroids but not all matroids in $\mathcal{G}^{\triangle}$.*

Excluded Minors. By considering some well-known matroids in $\mathcal{G}^{\square}$, $\mathcal{G}^{\triangle}$, and $\mathcal{G}^{\circ}$ we can get other interesting applications of Theorem 1.5. For example, for each $r \geq 2$, the whirl $\mathcal{W}^r$ is contained in $\mathcal{G}^{\triangle}$. Moreover, $\mathcal{G}^{\square}$ contains the Fano matroid F_7 and $\mathcal{G}^{\circ}$ contains the uniform matroid $U_{r,r+2}$. Thus we obtain the following result:

Corollary 1.9. *If $r \geq 2$ and $\mathcal{M}$ is the class of matroids with no minor isomorphic to $U_{2,r+2}, U_{r,r+2}, \mathcal{W}^r$ or F_7, then $h_{\mathcal{M}}(n) \approx \binom{n+1}{2}$.*

For each r, the free rank-r spike Λ_r is the truncation of $M(K_{2,r})$, so $\Lambda^r \in \mathcal{G}^{\circ}$, and $U_{r,r+2}$ can also be replaced by Λ_r in the above theorem.

For an odd-sized finite field $\mathrm{GF}(q)$ and $r \geq q$, all matroids but $\mathcal{W}^r$ in Corollary 1.9 are not $\mathrm{GF}(q)$-representable, giving something simpler:

Corollary 1.10. *If q is an odd prime power, $r \geq 2$ is an integer, and $\mathcal{M}$ is the class of $\mathrm{GF}(q)$-representable matroids with no $\mathcal{W}^r$-minor, then $h_{\mathcal{M}}(n) \approx \binom{n+1}{2}$.*

2. Preliminaries

We use the notation of Oxley [14]. A rank-1 flat is a *point* and a rank-2 flat is a *line*. Additionally, we write $|M|$ for $|E(M)|$ and $\varepsilon(M)$ for $|\operatorname{si}(M)|$, the number of points in M. For an integer $\ell \geq 2$, we write $\mathcal{U}(\ell)$ for the class of matroids with no $U_{2,\ell+2}$-minor.

We require a theorem of Kung [10] that bounds the number of points in a matroid in $\mathcal{U}(\ell)$.

Theorem 2.1. *If $\ell \geq 2$ and $M \in \mathcal{U}(\ell)$ then $\varepsilon(M) \leq \frac{\ell^{r(M)}-1}{\ell-1}$.*

We will use this theorem freely, usually with the weaker bound $\varepsilon(M) < \ell^{r(M)}$ for convenience of calculation. The next result we need is a constituent of the Growth Rate Theorem that shows that any matroid in $\mathcal{U}(\ell)$ with sufficiently large 'linear' density has a large clique as a minor.

Theorem 2.2. *There is a function $\alpha_{2.2} : \mathbb{Z}^2 \to \mathbb{R}$ so that, for all $n, \ell \in \mathbb{Z}^+$, if $M \in \mathcal{U}(\ell)$ and $\varepsilon(M) > \alpha_{2.2}(n,\ell)r(M)$, then M has an $M(K_{n+1})$-minor.*

We also need a special case of the Erdős-Stone theorem [3]:

Theorem 2.3. *There is a function $f_{2.3}(\alpha, m) : \mathbb{R} \times \mathbb{Z} \to \mathbb{Z}$ so that, for all $\alpha \in \mathbb{R}$, $n \in \mathbb{Z}$ with $\alpha > 0$ and $n \geq 1$, every simple graph G with $|V(G)| \geq f_{2.3}(\alpha, m)$ and $|E(G)| \geq \alpha |V(G)|^2$ has a $K_{m,m}$-subgraph.*

Finally, we require a version of Tutte's Linking Theorem proved by Geelen, Gerards and Whittle [5], for which we recall some standard notation. For disjoint sets $X, Y \subseteq E$ in a matroid $M = (E, r)$, we let $\lambda_M(X) = r(X) + r(E - X) - r(E)$ and we let $\kappa_M(X, Y)$ denote the minimum of $\lambda_M(Z)$ taken over all sets Z with $X \subseteq Z \subseteq E - Y$.

Theorem 2.4 (Tutte's Linking Theorem)**.** *If M is a matroid and $X, Y \subseteq E(M)$ are disjoint, then M has a minor N with ground set $X \cup Y$ so that $N|X = M|X$, $N|Y = M|Y$ and $\lambda_N(X) = \kappa_M(X, Y)$.*

3. Extensions of Cliques

In this section we prove Theorem 1.4. We first need some basic facts about extensions; all follow from material in [14], Section 7.2. A pair of flats F_1, F_2 of a matroid M is a *modular pair* in M if $r_M(F_1)+r_M(F_2) = r_M(F_1 \cup F_2)+r_M(F_1 \cap F_2)$. A flat is *modular* in M if it forms a modular pair with every flat of M. If $M \cong M(K_n)$, then a flat F of M is modular if and only if $M|F$ is connected.

We now consider extensions of cliques. Our first lemma deals with extensions where the new point is placed in some connected flat of rank much less than $r(M)$.

Lemma 3.1. *Let $m \geq 4$ be an integer. If M is a nongraphic extension of a clique by an element e, and $e \in \mathrm{cl}_M(F)$ for some modular flat F of $M \backslash e$ such that $r(M) - r_M(F) \geq m - 2$, then M has a minor isomorphic to $M_m^{\triangle}$ or $M_{m+1}^{\square}$.*

Proof. We may assume that M is minor-minimal subject to the hypotheses and let F be the minimal modular-flat of $M \backslash e$ with $e \in \mathrm{cl}_M(F)$. Let $r = r(M)$. Note that M is the modular sum (also known as *generalised parallel connection*) of $M \backslash e \cong M(K_{r+1})$ and $M|(F \cup \{e\})$, so M is uniquely determined by $M|(F \cup \{e\})$ and r.

By the minor-minimality of M, each element of F is on a line of length at least 3 with e. Since each pair of elements of $M(K_{r+1})$ is spanned by a modular flat of rank at most 3, we have that $r(F) \leq 3$.

Now it is easy to see that either $M|(F \cup \{e\}) \cong U_{2,4}$ (in which case $M \cong M^{\square}_m$) or $M|(F \cup \{e\}) \cong F_7$ (in which case $M \cong M^{\square}_{m+1}$). □

We now restate and prove Theorem 1.4.

Theorem 3.2. *Let m, n be integers such that $m \geq 4$ and $n \geq 2m^2$. If M is a nongraphic extension of a rank-n clique, then M has a minor isomorphic to M°_m, $M^{\triangle}_m$ or $M^{\square}_m$.*

Proof. Let $G \cong K_{n+1}$ and let $e \in E(M)$ be such that $M \backslash e \cong M(G)$. Let F be a minimal flat of $M \backslash e$ such that $e \in \mathrm{cl}_M(F)$. Since $M|F$ has at most $r_M(F)$ components and any two such components are joined by an edge of G, there is a flat $\hat{F}$ of M containing F such that $M|\hat{F}$ is connected and $r_M(\hat{F}) < 2r_M(F)$. If $r_M(\hat{F}) \leq 2m(m-1)$, then $n - r_M(\hat{F}) \geq m$, and M has a $M^{\triangle}_m$-minor or a $M^{\square}_m$-minor by Lemma 3.1. We may thus assume that $r_M(\hat{F}) > 2m(m-1)$ and so $r_M(F) > m(m-1)$.

Since F is a flat of $M(G)$ and $G \cong K_{n+1}$, there are vertex-disjoint complete subgraphs $C_1, C_2, \ldots, C_t$ of G such that $|V(C_i)| \geq 2$ for each i and $F = E(C_1) \cup \ldots \cup E(C_t)$; let $F_i = E(C_i)$ for each i. Note that $r_M(F) = \sum_{i=1}^{t} r_M(F_i)$. Let G' be the complete subgraph of G with vertex set $\cup_{i=1}^{t} V(C_i)$, so $r_M(E(G')) = r_M(F) + t - 1$.

If $r_M(F_i) \geq m-1$ for some i, then let B be a basis for F containing an $(m-1)$-element independent set $I \subseteq F_i$. Now $\mathrm{si}((M|F)/(B-I)) \cong M^{\circ}_m$, giving the lemma. Otherwise $r_M(F_i) < m-1$ for each i, so $r_M(F) < t(m-1)$. Therefore $t(m-1) > m(m-1)$ and $t > m$.

Let f be an edge of G' with one end in C_1 and the other in C_2. Let $M' = ((M|E(G'))/(\{f\} \cup (F - F_1 \cup F_2)))$. Let $F' = \mathrm{cl}_{M' \backslash e}(F_1 \cup F_2)$. Now $\mathrm{si}(M' \backslash e)$ is a clique. Moreover, $M'|F'$ is connected, has rank at least 2, and F' is a minimal flat of $M' \backslash e$ spanning e in M'. Since $r(M') = r(M|E(G')) - 1 - r_M(F) + r_M(F_1 \cup F_2) = r_M(F') + t - 2$ and $t > m$, Lemma 3.1 implies that $\mathrm{si}(M')$ has an $M^{\triangle}_m$-minor or an $M^{\square}_m$-minor. □

4. Complete Bipartite Graphs

In this section we show that a bounded lift of $M(K_{n,n})$ for very large n contains an $M(K_{m,m})$-restriction for some large m.

Lemma 4.1. *There is a function $f_{4.1} : \mathbb{Z}^2 \to \mathbb{Z}$ so that, for each $\ell, m, n \in \mathbb{Z}$ with $\ell \geq 2$, $m \geq 1$, and $n \geq f_{4.1}(\ell, m)$, if e is an element of a matroid $M \in \mathcal{U}(\ell)$ such that $M/e \cong M(K_{n,n})$, then $M \backslash e$ has a $K_{m,m}$-restriction.*

Proof. Set $f_{4.1}(\ell, m) = f_{2.3}\left(\frac{1}{8\ell+8}, m\right)$. Let $n \geq f_{4.1}(\ell, m)$, let $G \cong K_{n,n}$, and let e be an element of a matroid $M \in \mathcal{U}(\ell)$ such that $M/e =$

$M(G)$. Let T_1 and T_2 be vertex-disjoint copies of $K_{1,n-1}$ in G, let $T = E(T_1) \cup E(T_2)$, and let F denote the set of edges of G with an end in $V(T_1)$ and an end in $V(T_2)$. Note that $|F| > (n-1)^2 \geq \frac{n^2}{2}$ and that F is the set of nonloop elements of the rank-1 matroid $M/(\{e\} \cup T)$.

Now $M/T \in \mathcal{U}(\ell)$ and $r(M/T) \geq 2$, so there is some set $F' \subseteq F$ such that $|F'| \geq \frac{1}{\ell+1}|F|$ and F' is contained in a parallel class of M/T. Therefore F' has rank 1 in both M/T and $M/(T \cup \{e\})$, so $e \notin \mathrm{cl}_{M/T}(F')$ and $e \notin \mathrm{cl}_M(F')$. Thus $M|F' = (M/e)|F'$. But $G|F'$ is a simple graph with $2n$ vertices and at least $\frac{(n-1)^2}{\ell+1} = \frac{1}{8\ell+8}(2n)^2$ edges; since $n \geq f_{2.3}\left(\frac{1}{8\ell+8}, m\right)$, it follows by Theorem 2.3 that $G|F'$ has a $K_{m,m}$ subgraph, so $(M/e)|F' = M|F'$ has an $M(K_{m,m})$-restriction, as required. □

Lemma 4.2. *There is a function $f_{4.2} : \mathbb{Z}^3 \to \mathbb{Z}$ so that, for each $\ell, m, t, n \in \mathbb{Z}$ with $\ell \geq 2$, $m > t \geq 0$, and $n \geq f_{4.2}(\ell, m, t)$, if $M \in \mathcal{U}(\ell)$ and $C, X, K \subseteq E(M)$ satisfy $C \subseteq X$, $\sqcap_M(X, K) \leq t$ and $(M/C)|K \cong M(K_{n,n})$, then $M|(K - X)$ has an $M(K_{m,m})$-restriction.*

Proof. Let $\ell, m, t, n \in \mathbb{Z}$ with $\ell \geq 2$ and $m > t \geq 0$. Now let $m' = \max(m + t + 1, f_{4.1}(\ell, m))$ and define $f_{4.2}$ recursively by $f_{4.2}(\ell, m, t) = f_{4.2}(\ell, m', t-1)$.

Let $n \geq f_{4.2}(\ell, m, t)$, let $M \in \mathcal{U}(\ell)$, and let C, X, K be subsets of M such that $C \subseteq X$, $\sqcap_M(C, X) \leq t$ and $(M/C)|K \cong M(K_{n,n})$. We may assume that C is independent in M. Let C_1 be a maximal subset of C that is skew to K in M, and let $C_0 = C - C_1$. Now $(M/C_0)|K = M|K$ and $C_1 \subseteq \mathrm{cl}_{M/C_0}(K)$ by maximality. Moreover, $|C_1| \leq \sqcap_M(X, K) \leq t$. If $C_1 = \varnothing$ then $(M/C)|K = M|K \cong M(K_{n,n})$ and $r_M(X \cap K) \leq t$, so $M|K$ has an $M(K_{n-(t+1),n-(t+1)})$-restriction, giving the result since $n - (t+1) \geq m$. Otherwise, let $e \in C_1$ and let $M' = M/C_0$. Since $e \in X \cap \mathrm{cl}_{M'}(K)$, we have

$$\sqcap_{M'/e}(X - e, K) \leq \sqcap_{M'}(X, K) - 1 \leq t - 1.$$

Since $(M'/C_1)|K \cong M(K_{n,n})$ and $n \geq f_{4.2}(\ell, m', t-1)$, applying the inductive hypothesis to $C_1 - e, X - e$ and K in M'/e gives that $(M'/e)|(K - (X - e))$ has an $M(K_{m',m'})$-restriction R. By Lemma 4.1 applied to $M'|(\{e\} \cup E(R))$, the matroid $M'|E(R)$ has an $M(K_{m,m})$-restriction. Since $E(R) \subseteq K - X$ and $M'|E(R) = M|E(R)$, the lemma follows. □

5. Vertical Connectivity

We now detail a somewhat elaborate connectivity reduction, showing that quadratically dense classes contain dense, highly vertically connected matroids with some additional structure. We expect this reduction to be of much more general use in determining growth rate functions; we will invoke it in this paper just for $s = 4$.

Theorem 5.1. *Let $\mathcal{M}$ be a quadratically dense minor-closed class of matroids and let $p(x)$ be a real quadratic polynomial with positive leading coefficient. If $h_{\mathcal{M}}(n) > p(n)$ for infinitely many $n \in \mathbb{Z}^+$, then for all integers $r, s \geq 1$ there exists $M \in \mathcal{M}$ satisfying $\varepsilon(M) > p(r(M))$ and $r(M) \geq r$ such that either*

(1) M has an spanning clique restriction, or
(2) M is vertically s-connected and has an s-element independent set S so that $\varepsilon(M) - \varepsilon(M/e) > p(r(M)) - p(r(M) - 1)$ for each $e \in S$.

Proof. Let ℓ be an integer such that $U_{2,\ell+2} \notin \mathcal{M}$. Let $\mathcal{Q}$ be the set of all real quadratic polynomials q such that q has positive leading coefficient and $h_{\mathcal{M}}(n) > q(n)$ for infinitely many $n \in \mathbb{Z}^+$. Our first claim gives a weaker version of the theorem:

Claim 5.1.1. *For each $q \in \mathcal{Q}$ and $r, s \in \mathbb{Z}^+$, there is a matroid $M \in \mathcal{M}$ of rank at least r such that $\varepsilon(M) > q(r(M))$ and either*

(a) M has a spanning clique restriction, or
(b) M has an s-element independent set S such that each $e \in S$ satisfies $\varepsilon(M) - \varepsilon(M/e) > q(r(M)) - q(r(M) - 1)$.

Proof of claim: Let $n_2 \geq r + 1$ be an integer such that $q(x) - q(y) \geq \ell^s$ for all real x, y with $x \geq n_2$ and $x - 1 \geq y \geq 0$. Let $n_1 = (s(s-1)+1)n_2$. Let n_0 be an integer such that $q(x) \geq \alpha_{2.2}(n_1 - 1, \ell)x$ for all real $x \geq n_0$.

Let $M_0 \in \mathcal{M}$ satisfy $\varepsilon(M_0) > q(r(M_0))$ and $r(M_0) \geq n_0$. By Theorem 2.2 we know that M_0 has a $M(K_{n_1})$-minor K_0. Let M_1 be a minimal minor of M_0 such that $\varepsilon(M_1) > q(r(M_1))$ and N_0 is a minor of M_1. Note that $r(M_1) \geq r(N_0) \geq r$. Let C be an independent set in M_1 so that N_0 is a spanning restriction of M_1/C. By minimality, we have $\varepsilon(M_1) - \varepsilon(M_1/e) > q(r(M_1)) - q(r(M_1) - 1)$ for each $e \in C$. If $|C| \geq s$ then M_1 and C satisfy (b), so we may assume that $|C| < s$.

Let $i \geq 0$ be minimal so that there is a minor M_2 of M_1 for which

(i) $\varepsilon(M_2) > q(r(M_2))$, and
(ii) there exists $X \subseteq E(M_2)$ such that $r_{M_2}(X) \leq i$ and M_2/X has an $M(K_{(is+1)n_2})$-restriction N_2.

(Note that $i = s - 1$ and $M_2 = M_1$ is a candidate, so this choice is well-defined.) We consider two cases depending on whether $i = 0$.

Suppose that $i > 0$ and let $Y_1, Y_2, \ldots, Y_s, Z$ be mutually skew sets in N_2 so that $N_2|Y_i \cong M(K_{n_1})$ for each $i \in \{1, \ldots, s\}$ and $N_2|Z \cong$

$M(K_{((i-1)s+1)n_2})$; these sets can be chosen to correspond to vertex-disjoint cliques in the clique underlying N_2. If $M_2|Y_j = N_2|Y_j$ for some $j \in \{1,\dots,s\}$, then M_2 has an $M(K_{r+1})$-restriction so satisfies (i) and (ii) for $i = 0$, contradicting the mimimality of i. Thus, $M_2|Y_j \neq N_2|Y_j$ for each j, implying that $\sqcap_{M_2}(Y_j, X) > 0$ and $r_{M_2/Y_j}(X) \leq r_{M_2}(X) - 1 \leq i-1$ for each j. Let $Y = Y_1 \cup \ldots \cup Y_s$ and let J be a maximal subset of Y such that $\varepsilon(M_2/J) > q(r(M_2/J))$. Let $M_3 = M_2/J$. If $Y_j \subseteq J$ for some j, then $r_{M_3}(X) \leq i-1$ and $(M_3/X)|Z = N_2|Z \cong M(K_{((i-1)s+1)n_2})$, contradicting the minimality of i. Therefore $Y - J$ contains a transversal T of $(Y_1,\dots,Y_s)$. T is an s-element independent set of N_2/J and therefore of M_2/J. Moreover, by maximality of J, each $e \in T$ satisfies $\varepsilon(M_3) - \varepsilon(M_3/e) > q(r(M_3)) - q(r(M_3)-1)$. Since $r(M_3) \geq r(N_2|Z) \geq n_2 - 1 \geq r$, now (b) holds for M_3 and T.

Now suppose that $i = 0$. Then N_2 is an $M(K_{r+1})$-restriction of M_2. Let M_4 be a minimal minor of M_2 such that $\varepsilon(M_4) > q(r(M_4))$ and N_2 is a restriction of M_4. If N_2 is spanning in M_4 then (a) holds. Otherwise, by minimality we have $\varepsilon(M_4|\operatorname{cl}_{M_4}(E(N_2))) \leq q(r(N_2))$, so since $r(M_4) \geq n_2$ we have

$$\begin{aligned}\varepsilon(M_4\backslash \operatorname{cl}_{M_4}(E(N_2))) &> q(r(M_4)) - q(r(N_2))\\ &\geq q(r(M_4)) - q(r(M_4)-1)\\ &\geq \ell^s.\end{aligned}$$

Therefore there is an s-element independent set S of M_4 that is disjoint from $\operatorname{cl}_{M_4}(E(N_2))$. Since N_2 is a restriction of M_4/e for each $e \in S$, it follows that M_4 and S satisfy (b). □

Suppose that the theorem does not hold for some positive integers s_0 and r_0. Let $a, b, c \in \mathbb{R}$ such that $p(x) = ax^2 + bx + c$; thus $a > 0$.

Claim 5.1.2. *The quadratic polynomial $p(x)+\nu x$ is in $\mathcal{Q}$ for all $\nu \in \mathbb{R}$.*

Proof of claim: Suppose not; then there exists some $\nu \geq 0$ for which $p(x) + \nu x \in \mathcal{Q}$ but $p(x) + (\nu + a)x \notin \mathcal{Q}$. Let r_1 be an integer so that

$$(2s_0 + 1)a(x + y) + s_0|\nu + b| + c - as_0^2 \leq 2axy \tag{1}$$

for all real $x, y \geq r_1$, and

$$h_{\mathcal{M}}(n) \leq p(n) + (\nu + a)n \text{ for every integer } n \geq r_1. \tag{2}$$

Let $r_2 \geq \max(r_0, 2r_1)$ be an integer so that

$$p(x) - p(x-1) > ax + \ell^{r_1} \text{ for all real } x \geq r_2. \tag{3}$$

By the first claim, there exists $M \in \mathcal{M}$ of rank at least r_2, such that $\varepsilon(M) > p(r(M)) + \nu r(M)$ and either M has a spanning clique or there is an s_0-element independent set S of M so that

$$\varepsilon(M) - \varepsilon(M/e) > p(r(M)) - p(r(M) - 1) + \nu$$

for each $e \in S$. Since $\nu \geq 0$ and the theorem does not hold for s_0 and r_0, the matroid M is not vertically s_0-connected. We may assume that M is simple; let (A, B) be a partition of $E(M)$ so that $r_M(A) \leq r_M(B) < r(M)$ and $r_M(A)+r_M(B)-r(M) < s_0-1$. Let $r = r(M), r_A = r_M(A)$ and $r_B = r_M(B)$.

If $r_A < r_1$, then $|A| < \ell^{r_1}$, so since $r \geq r_2$, by (3) we have

$$\begin{aligned} |B| = |M| - |A| &> p(r) + \nu r - \ell^{r_1} \\ &> p(r-1) + (\nu + a)r \\ &\geq p(r_B) + (\nu + a)r_B, \end{aligned}$$

contradicting (2), since $r_B \geq r - r_A \geq r_2 - r_1 \geq r_1$. So we have $r_B \geq r_A \geq r_1$. Therefore, using (2) we have

$$p(r) + \nu r < |A| + |B| \leq p(r_A) + p(r_B) + (\nu + a)(r_A + r_B).$$

Using $r_A + r_B < r + s_0$, expanding $p(x) = ax^2 + bx + c$ and simplifying, we have

$$(2s_0 + 1)a(r_A + r_B) + s_0|\nu + b| + c - as_0^2 > 2ar_Ar_B.$$

Since $r_B \geq r_A \geq r_1$, this contradicts (1). $\square$

Let $\alpha > 0$ be such that $h_{\mathcal{M}}(n) \leq \alpha p(n)$ for all $n \in \mathbb{Z}^+$. Let n_1 be an integer so that $p(x) \geq p(x-1) \geq 0$ for all real $x \geq n_1$ and

$$a(\alpha + 2s_0)(x + y) + ((\alpha + 1)b + \alpha|c|)s_0 + c - as_0^2 \leq 2axy$$

for all real $x, y \geq n_1$. Let $\nu = \max(-b, \ell^{n_1}, \ell^{n_1} - \min_{x\in\mathbb{R}} p(x))$.

Let $M \in \mathcal{M}$ be minor-minimal such that $r(M) > 0$ and $\varepsilon(M) > p(r(M))+\nu r(M)$. (Such a matroid exists by the previous claim.) Note that M is simple; let $r = r(M)$. We have $\varepsilon(M) > \nu + p(r(M)) \geq \ell^{n_1}$, so $r(M) \geq n_1$.

For each $e \in E(M)$, minimality of M implies that

$$\varepsilon(M) - \varepsilon(M/e) > p(r) - p(r-1) + \nu.$$

This expression exceeds $p(r) - p(r-1)$, and $r(M) \geq n_1 \geq \max(r_0, s_0)$; since the lemma does not hold for s_0 and r_0, we know that M is not vertically s_0-connected. Let (A, B) be a partition of $E(M)$ so that $r_M(A) \leq r_M(B) < r$ and $r_M(A) + r_M(B) < r(M) + s_0 - 1$. Let $r_A = r_M(A)$, $r_B = r_M(B)$.

We first argue that $r_A \geq n_1$. If not, then $|A| < \ell^n$, so we have

$$\begin{aligned}|B| &= |M| - |A| \\ &> p(r) + \nu r - \ell^{n_1} \\ &\geq p(r-1) + \nu(r-1) \\ &\geq p(r_B) + \nu r_B,\end{aligned}$$

which contradicts minimality. Next, since $r \geq n_1$ we have $p(r) \geq 0$ and so $\nu r < |M| \leq \alpha p(r)$; since $r \geq 1$ this implies that

$$\nu \leq \alpha(ar + b + \tfrac{c}{r}) \leq \alpha(a(r_A + r_B) + b + |c|).$$

Now

$$p(r_A) + \nu r_A + p(r_B) + \nu r_B \geq |M| > p(r) + \nu r.$$

Using $r_A + r_B < r + s_0$ and $\nu + b \geq 0$, expanding p as earlier gives

$$s_0(\nu + b) + c - as_0^2 + 2as_0(r_A + r_B) > 2r_A r_B.$$

Combining this with our estimate for ν, we have

$$a(\alpha + 2s_0)(r_A + r_B) + ((\alpha+1)b + \alpha|c|)s_0 + c - as_0^2 > 2ar_A r_B,$$

contradicting $r_B \geq r_A \geq n_1$ and the definition of n_1. □

6. Spikes

A point of a matroid M whose contraction substantially reduces the number of points of M often gives rise to a *spike*. This structure is well-known and its definitions vary slightly across the literature; here we give a definition convenient for extremal arguments that allows for any positive number of 'tips' but no 'co-tips'.

A *spike* is a matroid S with ground set $E(S) = X \cup Y \cup T$, where X, Y, T are disjoint sets so that T is a nonempty parallel class, $S|(X \cup Y)$ is simple, and X and Y are circuits of S/T so that each line of S containing T contains exactly one element of each of X and Y. Note that $|X| = |Y|$. An element in T is a *tip* of S.

It is clear from this definition that if $r(S) \geq 2$ then contracting a non-tip element yields a rank-$(r(S)-1)$ spike. If $r(S) = 3$ then S has three distinct three-point lines through its tip, so $\varepsilon(S) = 7$ and thus S is nongraphic; therefore all spikes of rank at least three are nongraphic.

Lemma 6.1. *If S is a spike-restriction of a matroid M, and $e \in E(M)$ is not parallel to a tip of S, then there are spike-restrictions S_1 and S_2 of M/e such that $E(S) - \{e\} = E(S_1) \cup E(S_2)$.*

Proof. If $e \notin \mathrm{cl}_M(E(S))$ or e is parallel to an element of $E(S)$, then the result holds with $S_1 = S_2 = S$, so we may assume otherwise; we may also assume that $E(M) = E(S) \cup \{e\}$. Let T, X, Y be sets as in the definition, and let $t \in T$. It suffices to show that $(M/\{t, e\})|X$ is the union of two circuits. Since X is a circuit of M/t, we have $r_{(M/t)^*}(X) = 1$, so $r_{(M/t)^*}(X \cup \{e\}) \leq 2$ and so $r^*(M/\{t, e\}|X) \leq 2$. Every matroid of rank at most 2 is clearly the union of two cocircuits, so $(M/\{t, e\})|X$ is the union of two circuits, as required. □

Lemma 6.2. *Let S be a spike-restriction of a matroid M. If R is a restriction of $M \backslash E(S)$ satisfying $\kappa_M(E(S), E(R)) \geq 3$, then M has a minor with R as a spanning restriction and with a nongraphic spike-restriction.*

Proof. Let M' be a minimal minor of M such that R is a restriction of M', and $M' \setminus E(R)$ has a spike-restriction S' such that $\kappa_{M'}(E(R), E(S')) \geq 3$. By Theorem 2.4, we have $E(M') = E(R) \cup E(S')$. Contracting any non-tip element of S' that is not in $\mathrm{cl}_{M'}(E(R))$ gives a minor that contradicts the minimality of M', so every non-tip element of S' is spanned by $E(R)$. Since S' has no coloops, it follows that R is spanning in M', giving the result □

We use the above lemma to show that a matroid with a spike-restriction with sufficient connectivity to a large complete bipartite graph has a large nongraphic extension of a clique as a minor:

Lemma 6.3. *Let $m \geq 3$ be an integer. If M is a matroid with a spike-restriction S, and $M \backslash E(S)$ has an $M(K_{m+3,m+3})$-restriction R so that $\kappa_M(E(R), E(S)) \geq 3$, then M has a minor isomorphic to a nongraphic extension of $M(K_{m+1})$.*

Proof. By Lemma 6.2, there is a minor M_1 of M with R as a spanning restriction and with a spike-restriction of rank at least 3. Let $H \cong K_{m+3,m+3}$ be such that $R = M(H)$. Let J be a matching of H that is maximal so that $|J| \leq m$ and M_1/J has a spike-restriction S of rank at least 3.

If $|J| = m$, then H/J has a K_{m+1}-subgraph and is clearly 4-connected. Therefore $M(H)/J$ is a spanning vertically 4-connected restriction of M_1/J with an $M(K_{m+1})$-restriction R'. By vertical 4-connectivity we have $\kappa_{M_1/J}(E(R'), E(S)) \geq 3$, so by Lemma 6.2 there is a minor M_2 of M_1/J with R' as a spanning restriction and with a nongraphic spike-restriction; this contains a nongraphic extension of R', giving the lemma.

If $|J| < m$, then there are at least 8 vertices of H unsaturated by J, so there is a 6-element independent set $I \subseteq E(H) - J$ such that $J \cup \{f\}$ is a matching for each $f \in I$. By maximality, we have $f \in \mathrm{cl}_{M_1/J}(E(S))$ for each $f \in I$, so $r(S) \geq 6$. Let $e \in I$ be not parallel to a tip of S in M_1/J. By Lemma 6.1, there are spike-restrictions S_1, S_2 of $M_1/(J \cup \{e\})$ such that $E(S_1) \cup E(S_2) = E(S) - \{e\}$. But $E(S) - \{e\}$ has rank at least 5 in $M_1/(J \cup \{e\})$, so S_1 or S_2 has rank at least 3, contradicting the maximality of J. □

7. Tangles

In this section we discuss tangles, structures that capture the idea of connectivity into a minor. Tangles were introduced for graphs, and implicitly for matroids, by Robertson and Seymour [15] and were later extended explicitly to matroids [2,4]. The material in this section follows [7] and [13].

Let M be a matroid and let $\theta \in \mathbb{Z}^+$. A set $X \subseteq E(M)$ is *k-separating in M* if $\lambda_M(X) < k$. A collection $\mathcal{T}$ of subsets of $E(M)$ is a *tangle of order θ* if

(1) Every set in T is $(\theta - 1)$-separating in M and, for each $(\theta-1)$-separating set $X \subseteq E(M)$, either $X \in T$ or $E(M) - X \in \mathcal{T}$;
(2) if $A, B, C \in \mathcal{T}$ then $A \cup B \cup C \neq E(M)$; and
(3) $E(M) - \{e\} \notin \mathcal{T}$ for each $e \in E(M)$.

We refer to the sets in $\mathcal{T}$ as *$\mathcal{T}$-small.* Given a tangle of order θ on a matroid M and a set $X \subseteq E(M)$, we set $\kappa_{\mathcal{T}}(X) = \theta - 1$ if X is contained in no $\mathcal{T}$-small set, and $\kappa_{\mathcal{T}}(X) = \min\{\lambda_M(Z) : X \subseteq Z \in \mathcal{T}\}$ otherwise. The proof of our first lemma appears in [4]:

Lemma 7.1. *If $\mathcal{T}$ is a tangle of order θ on a matroid M, then $\kappa_{\mathcal{T}}$ is the rank function of a rank-$(\theta - 1)$ matroid on $E(M)$.*

This matroid, which we denote $M(\mathcal{T})$, is the *tangle matroid.* The next lemma is easily proved:

Lemma 7.2. *If N is a minor of a matroid M and $\mathcal{T}_N$ is a tangle of order θ on N, then $\{X \subseteq E(M) : \lambda_M(X) < \theta - 1, X \cap E(N) \in \mathcal{T}_N\}$ is a tangle of order θ on M.*

This is the tangle on M *induced* by $\mathcal{T}_N$.

If M is a matroid and k is an integer, then we write $\mathcal{T}_k(M)$ for the collection of $(k-1)$-separating sets of M that are neither spanning nor cospanning. For example, if $M \cong M(K_{n+1})$ and $k = \lceil 2n/3 \rceil$, then $\mathcal{T}_k(M)$ is simply the collection of subsets of $E(M)$ of rank at most $k-2$. Since K_{n+1} is not the union of three subgraphs on at most $\frac{2}{3}n$ vertices, we easily have the following:

Lemma 7.3. *If $n \geq 2$ and $M \cong M(K_{n+1})$, then $\mathcal{T}_{\lceil 2n/3 \rceil}(M)$ is a tangle of order $\lceil 2n/3 \rceil$ in M.*

If M is a matroid with an $M(K_{n+1})$-minor N, then we write $\mathcal{T}_{\lceil 2n/3 \rceil}(M, N)$ for the tangle of order n in M induced by $\mathcal{T}_{\lceil 2n/3 \rceil}(N)$.

The next result is a slight variation of a lemma from [7].

Lemma 7.4. *Let $k \in \mathbb{Z}^+$, let M be a matroid and let N be a minor of M such that $\mathcal{T}_k(N)$ is a tangle. If $X \subseteq E(M)$ is contained in a $\mathcal{T}_k(M, N)$-small set, then there is a minor M' of M such that $M'|X = M|X$, M' has N as a minor, and X is contained in a $\mathcal{T}_k(M', N)$-small set X' such that $E(M') = E(N) \cup X'$ and $\lambda_{M'}(X') = \kappa_{\mathcal{T}_k(M',N)}(X) = \kappa_{\mathcal{T}_k(M,N)}(X)$.*

Proof. Let $b = r_{\mathcal{T}_k(M,N)}(X)$ and let M' be a minimal minor of M such that N is a minor of M, $M|X = M'|X$ and $r_{\mathcal{T}_k(M',N)}(X) = b$. Let $\mathcal{T} = \mathcal{T}_k(M', N)$ and $X' = \mathrm{cl}_{M(\mathcal{T})}(X)$. It remains to show that $E(M') = X' \cup E(N)$. If not, there is some $e \in E(M') - X' \cup E(N)$. Since $\mathrm{cl}_{M'}(X) \subseteq X'$, we know that $M|X$ is a restriction of both M/e and $M \backslash e$. If N is a minor of M/e, and so by choice of M we have $r_{\mathcal{T}_k(M/e,N)}(X) \leq b-1$. Therefore there is some set $Z \in \mathcal{T}_k(M/e, N)$ such that $\lambda_{M'/e}(Z) \leq b-1$ and $X \subseteq Z$. Therefore $Z \cup \{e\} \in \mathcal{T}$ and $\lambda_{M'}(Z \cup \{e\}) \leq b$ so $r_{\mathcal{T}}(X \cup \{e\}) = r_{\mathcal{T}}(X)$ and $e \in \mathrm{cl}_{\mathcal{T}}(X)$, a contradiction. The case where N is a minor of $M \backslash e$ is similar. $\square$

The next lemma is our main technical application of tangles; it shows that a restriction X of a matroid M with a huge clique minor can be contracted onto a large clique restriction with as much connectivity as could be expected:

Lemma 7.5. *There is a function $f_{7.5} : \mathbb{Z}^2 \to \mathbb{Z}$ so that, for all $m, n, \ell \in \mathbb{Z}$ with $m > 0$, $\ell \geq 2$ and $n \geq f_{7.5}(m, \ell)$, if $M \in \mathcal{U}(\ell)$ has an $M(K_{n+1})$-minor N with corresponding tangle $\mathcal{T} = \mathcal{T}_{\lceil 2n/3 \rceil}(M, N)$ and $X \subseteq E(M)$ satisfies $\kappa_{\mathcal{T}}(X) \leq m$, then M has a minor M' with an $M(K_{m+1})$-restriction R so that $X \cap E(R) = \varnothing$, $M'|X = M|X$, $E(M') = E(R) \cup X$ and $\lambda_{M'}(X) = \kappa_{\mathcal{T}}(X)$.*

Proof. Let $n_1 = f_{4.2}(\ell, m, m)$ and let $n = \max(2m, 2n_1 - 1)$.

Let $t = r_{\mathcal{T}}(X)$ and $k = \lceil 2n/3 \rceil$. Note that $t \leq m < k$. Since $r_{\mathcal{T}}(X) = t$, the set X is contained in a $\mathcal{T}$-small set. By Lemma 7.4, there is a minor M_1 of M such that $M_1|X = M|X$, M_1 has N as a minor, and X is contained in a $\mathcal{T}_k(M_1, N)$-small set X' such that $E(M_1) = E(N) \cup X'$ and $\lambda_{M_1}(X') = r_{\mathcal{T}_k(M_1,N)}(X) = r_{\mathcal{T}}(X) = t$. Since $N \cong M(K_{n+1})$ and $X' \cap E(N)$ is $\mathcal{T}_k(N)$-small, it follows that $r(M_1|(E(N) - X')) = r(M_1|E(N))$ and so we also have $\sqcap_{M_1}(X', E(N)) = t$.

Let $C \subseteq E(M_1)$ be such that N is a restriction of M_1/C. Let N' be an $M(K_{n_1,n_1})$-restriction of N. Since $E(N') \subseteq E(N)$, we have $\sqcap_{M_1}(X', E(N')) \leq \sqcap_{M_1}(X', E(N)) = t$. By Lemma 4.2, we see that $M_1|(E(N') - X')$ has an $M(K_{m,m})$-restriction R'. Note that $X \cap E(R') = \varnothing$ and $\kappa_{M_1}(X, E(R')) \leq \lambda_{M_1}(X') \leq t$. Moreover we have $r(R') = 2m - 1 > t$, so, since $r_{\mathcal{T}_k(M_1,E(N))}(X) = t$, we must have $\kappa_{M_1}(X, E(R')) = t$, as otherwise M_1 has a t-separation for which neither side is $\mathcal{T}_k(M_1, N)$-small.

By Theorem 2.4, the matroid M_1 has a minor M_2 such that $E(M_2) = X \cup E(R')$, $M_2|X = M_1|X, M_2|E(R') = R'$, and $\lambda_{M_2}(X) = t$. Let $R = M(H)$, where $H \cong K_{m(m+1),m(m+1)}$, and let $H_1, \dots, H_{m+1}$ be vertex-disjoint $K_{m,m}$-subgraphs of H. Now the sets $E(H_i)$ are mutually skew in M_2, so $\sum_{i=1}^{m+1} \sqcap_{M_2}(X, E(H_i)) \leq \sqcap_{M_2}(X, E(H)) = t \leq m$, so there is some i such that $\sqcap_{M_2}(X, E(H_i)) = 0$. Let J be the edge set of an $(m-1)$-edge matching of H_i and let $M_3 = M_2/J$. Now $M_3|(H_i - J)$ has a K_{m+1}-restriction R, and $\lambda_{M_3}(X) = \lambda_{M_2}(X) = t$.

Let B be a basis for M_3 containing a basis B' for $M_3 \setminus X$. Note that $M_3/(B - B')$ has $M(H/J)$ as a spanning restriction and H/J is an $(m + 1)$-connected graph, so $M_3/(B - B')$ is vertically $(m + 1)$-connected. Since $B - B'$ is skew to $E(M_3 \setminus X)$, we have

$$\begin{aligned}
\kappa_{M_3}(X, E(R)) &= \kappa_{M_3/(B-B')}(X - (B - B'), E(R)) \\
&\geq \min(m, r_{M_3/(B-B')}(X - (B - B')), r_{M_3/(B-B')}(E(R))) \\
&= \min(t, m, m) = t.
\end{aligned}$$

Theorem 2.4 now gives the required minor. $\square$

When M is vertically $(t+1)$-connected and $r_M(X) \leq t$ in the above lemma, we have $\kappa_{\mathcal{T}}(X) = r_M(X)$, and we obtain a simpler corollary:

Corollary 7.6. *There is a function $f_{7.6} : \mathbb{Z}^2 \to \mathbb{Z}$ so that, for all $t, m, n, \ell \in \mathbb{Z}$ with $m \geq t > 0$, $\ell \geq 2$ and $n \geq f_{7.6}(m, \ell)$, if $M \in \mathcal{U}(\ell)$ is a vertically $(t + 1)$-connected matroid with an $M(K_{n+1})$-minor and $X \subseteq E(M)$ satisfies $r_M(X) \leq t$, then M has a rank-m minor N with an $M(K_{m+1})$-restriction such that $X \subseteq E(N)$ and $N|X = M|X$.*

8. The main result

We can now prove our main theorem. First we show that a spike with connectivity 3 to a huge clique minor gives a nongraphic extension of a large clique in a minor:

Lemma 8.1. *There is a function $f_{8.1} : \mathbb{Z}^2 \to \mathbb{Z}$ so that, for each $m, \ell, n \in \mathbb{Z}$ with $m \geq 3$, $\ell \geq 2$, and $n \geq f_{8.1}(m, \ell)$, if $M \in \mathcal{U}(\ell)$ is a matroid with an $M(K_{n+1})$-minor N and a spike-restriction whose* *ground set has connectivity at least* 3 *to the tangle* $\mathcal{T}_{\lceil 2n/3\rceil}(M,N)$*, then* M *has a minor isomorphic to a nongraphic extension of* $M(K_{m+1})$*.*

Proof. Let $m \geq 3$ and $\ell \geq 2$ be integers. Let $n' = f_{4.1}(\ell, m+3)$. Set $f_{8.1}(m,\ell) = \max(2n', f_{7.5}(\ell,m))$.

Let $n \geq f_{8.1}(m,\ell)$ and let $k = \lceil 2n/3\rceil$. Let $M \in \mathcal{U}(\ell)$ be a matroid with an $M(K_{n+1})$-minor N and a spike-restriction S_0 such that $\kappa_{\mathcal{T}_k(M,N)}(E(S_0)) \geq 3$. We show that M has a nongraphic extension of $M(K_{m+1})$ as a minor; by considering a parallel extension of M if necessary, we may assume that $E(S_0) \cap E(N) = \varnothing$. Let M_1 be a minimal minor of M such that

(1) N is a minor of M_1, and
(2) $M_1 \backslash E(N)$ has a spike-restriction S such that $\kappa_{\mathcal{T}_k(M_1,N)}(E(S)) \geq 3$.

Let C be an independent set in M_1 such that N is a spanning restriction of M_1/C. If $|C| \leq 1$ then $N = (M_1/C)|E(N)$ has an $M(K_{n',n'})$-restriction, so by Lemma 4.1 the matroid $M_1|E(N)$ has an $M(K_{m+3,m+3})$-restriction R_1. Moreover, we clearly have $\kappa_{\mathcal{T}_k(M_1,N)}(E(R_1)) \geq 2(m+3)-1 \geq 3$, so $\kappa_{M_1}(E(S),E(R_1)) \geq 3$, as otherwise we have a (≤ 3)-separation with both sides $\mathcal{T}_k(M_1,N)$-small. By Lemma 6.3, the result holds.

If $|C| \geq 2$ then there is some $e \in C$ that is not parallel in M to a tip of S. By Lemma 6.1, there are spike-restrictions S_1, S_2 of M_1/e such that $E(S_1) \cup E(S_2) = E(S)$. By minimality of M_1, we have $\kappa_{\mathcal{T}_k(M_1/e,N)}(E(S_i)) \leq 2$ for each $i \in \{1,2\}$. It follows since $\kappa_{\mathcal{T}_k(M_1/e,N)}$ is the rank function of a matroid on M_1/e that $\kappa_{\mathcal{T}_k(M_1/e,N)}(E(S)) \leq 2+2 = 4$ and so $\kappa_{\mathcal{T}_k(M_1,N)}(E(S)) \leq 5$.

By Lemma 7.5 and the definition of n, there is a minor M_2 of M_1 with an $M(K_{m+1})$-restriction R_2 such that $E(R_2) \cap E(S) = \varnothing$, $E(M_2) = E(R_2) \cup E(S)$, $3 \leq \lambda_{M_2}(E(S)) \leq 5$ and $S = M_2|E(S)$. Since $\kappa_{M_2}(E(S),E(R_2)) = \lambda_{M_2}(E(S)) \geq 3$, Lemma 6.2 implies that M_2 has a minor with R_2 as a spanning restriction and with a nongraphic spike-restriction. The result follows. $\square$

Finally, we restate and prove Theorem 1.1.

Theorem 8.2. *Let* $m \geq 3$ *and* $\ell \geq 2$ *be integers. If* $\mathcal{M}$ *is the class of matroids with no* $U_{2,\ell+2}$*-minor and with no nongraphic extension of* $M(K_{m+1})$ *as a minor, then* $h_{\mathcal{M}}(n) \approx \binom{n+1}{2}$*.*

Proof. Suppose that the theorem fails. Clearly $\mathcal{M}$ contains the graphic matroids, so $h_{\mathcal{M}}(n) \geq \binom{n+1}{2}$ for all n; thus, we have $h_{\mathcal{M}}(n) > \binom{n+1}{2}$ for infinitely many n.

Let $n_0 = \max(f_{7.6}(m,\ell), f_{8.1}(m,\ell))$ and $n_1 = \max(m, 2\alpha_{2.2}(n_0,\ell))$. By Theorem 5.1 with $p(x) = \binom{x+1}{2}$, $s = 4$ and $r = n_1$, we see that there exists $M \in \mathcal{M}$ such that $r(M) \geq n_1$, $\varepsilon(M) > \binom{r(M)+1}{2}$ and either

(1) M has a spanning clique, or
(2) M is vertically 4-connected and there is some nonloop e of M such that $\varepsilon(M) - \varepsilon(M/e) > r(M)$.

We may assume that M is simple. If (1) holds, then since $|M| > \binom{r(M)+1}{2}$, the matroid M has a nongraphic extension of a rank-$r(M)$ clique as a restriction. Since $r(M') \geq n_1 \geq m \geq 3$, it is easy to repeatedly contract elements of M' and simplify to obtain a nongraphic extension of $M(K_{m+1})$, a contradiction. Therefore (2) holds.

Now $r(M) \geq 2\alpha_{2.2}(n_0,\ell)$, so $\varepsilon(M) > \binom{r(M)+1}{2} > \alpha_{2.2}(n_0,\ell)r(M)$; thus, M has an $M(K_{n_0+1})$-minor N by Theorem 2.2.

Let $\mathcal{L}$ be the set of lines of M containing e. If $|L| \geq 4$ for some $L \in \mathcal{L}$, then by vertical 3-connectivity of M, Corollary 7.6 implies that M has a rank-m minor M' with an $M(K_{m+1})$-restriction such that $M'|L = M|L$. Since $M'|L$ is nongraphic, this minor contains a nongraphic extension of $M(K_{m+1})$, a contradiction. So $|L| \leq 3$ for each $L \in \mathcal{L}$, and each parallel class of M/e has size 1 or 2.

Let $\mathcal{L}_3 = \{L \in \mathcal{L} : |L| = 3\}$. Note that $r(M) < \varepsilon(M) - \varepsilon(M/e) = 1 + |\mathcal{L}_3|$, so $r(M) \leq |\mathcal{L}_3|$. Therefore there are at least $r(M) > r(M/e)$ parallel pairs in M/e, so there is a circuit C of M/e such that $|C| \geq 3$ and each $x \in C$ lies in a parallel class of size 2 in M/e. Therefore e is the tip of a nongraphic spike-restriction S of M. Since M is vertically 4-connected, the set $E(S)$ has rank at least 3 in the tangle $\mathcal{T}_{\lceil 2n_0/3 \rceil}(M,N)$. By the definition of n_0, Lemma 8.1 gives a nongraphic extension of $M(K_{m+1})$ as a minor of M, again a contradiction. $\square$

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