A splitter theorem for internally 4-connected graphs and binary matroids

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Connectivity
Connectivity

\[ \oplus \]

\[ \oplus ^2 \]

\[ = \]

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Connectivity

\[
\begin{align*}
\begin{array}{ccc}
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\end{array}
\end{align*}
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\begin{align*}
\begin{array}{ccc}
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\end{align*}
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\begin{align*}
\begin{array}{ccc}
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\begin{array}{ccc}
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\end{array}
\end{align*}
\]
Connected matroids

Theorem (Tutte, 1966)

$M$ 2-connected $\implies M \setminus e$ or $M / e$ is 2-connected for all $e \in E(M)$
Connected matroids

Theorem (Tutte, 1966)
$M$ 2-connected $\implies M\setminus e$ or $M/e$ is 2-connected for all $e \in E(M)$

Theorem (Tutte, 1966)
$M$ 3-connected $\implies \exists M' \leq M$ where $M'$ is 3-connected and

$|E(M)| - |E(M')| = 1$
Connected matroids

Theorem (Tutte, 1966)

\[ M \text{ 2-connected} \implies \exists e \in E(M) \] with \( M\setminus e \) or \( M/e \) is 2-connected for all \( e \in E(M) \)

Theorem (Tutte, 1966)

\[ M \text{ 3-connected} \implies \exists M' \preceq M \text{ where } M' \text{ is 3-connected and} \]
\[ |E(M)| - |E(M')| = 1 \text{ unless} \]
Connected matroids

Theorem (Tutte, 1966)

$M$ 2-connected $\implies M \setminus e$ or $M / e$ is 2-connected for all $e \in E(M)$

Theorem (Tutte, 1966)

$M$ 3-connected $\implies \exists M' \preceq M$ where $M'$ is 3-connected and

$$|E(M)| - |E(M')| = 1 \text{ unless } M \text{ is a wheel or whirl}$$
Connectivity

\[ \oplus \]

\[ \oplus_2 \]

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Connectivity

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Connectivity

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Connectivity

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\[ \oplus_3 \]
Connectivity

internally 4-connected violator, AKA 4-fan
Connectivity

internally 4-connected violator, AKA 4-fan
Connectivity

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Connectivity

internally 4-connected violator, AKA 4-fan
Theorem (2011)

Let $M$ be an internally 4-connected binary matroid. Then $M$ has an internally 4-connected minor $M'$ with

$$1 \leq |E(M) - E(M')| \leq 3$$

unless

(i) a terrahawk; or (ii) a planar or Möbius quartic ladder.
Theorem (2011)

Let $M$ be an internally 4-connected binary matroid. Then $M$ has an internally 4-connected minor $M'$ with

$$1 \leq |E(M) - E(M')| \leq 3$$

unless $M$ or $M^*$ is the cycle matroid of

(i) a terrahawk; or (ii) a planar or Möbius quartic ladder.
Theorem (Brylawski 1972, Seymour 1977)

$M$, $N$ 2-connected, $N \preceq M$, and $e \in E(M) - E(N) \implies M \setminus e$ or $M / e$ is 2-connected with $N$ as a minor.
3-connected matroids

Theorem (Seymour’s Splitter Theorem)

$M$ & $N$ 3-connected and $N \not\subseteq M$, where $|E(N)| \geq 4$ and, if $N$ is a wheel/whirl, then $M$ has no larger wheel/whirl-minor $\implies \exists M'$ where $M' \not\subseteq M$ and $N \leq M'$ and $|E(M)| - |E(M')| = 1$
Internally 4-connected binary matroids

Graphs are binary matroids, what do we know for graphs?

Johnson and Thomas (2001)
Internally 4-connected binary matroids

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Johnson and Thomas (2001)

What about binary matroids?
Geelen and Zhou (2006)
Zhou (2012)
Internally 4-connected binary matroids

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These allow the intermediate matroid to satisfy some weaker form of connectivity.
Internally 4-connected binary matroids

Graphs are binary matroids, what do we know for graphs?
Johnson and Thomas (2001)

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These allow the intermediate matroid to satisfy some weaker form of connectivity.

What did Johnson and Thomas show, exactly?
Splitter Theorem for internally 4-connected graphs
(Generating internally 4-connected graphs, JCTB 2001)
Thor Johnson and Robin Thomas

CPL\textsubscript{k}  

CML\textsubscript{k}  

CPB-W\textsubscript{k}  

CMB-W\textsubscript{k}
Theorem (Theorem 1)

$G$ & $H$ internally 4-connected graphs and $H \nsubseteq G$ and, if $H$ is a $CPL_k$
Theorem (Theorem 1)

$G$ & $H$ internally 4-connected graphs and $H \not\preceq G$ and, if $H$ is a $CPL_k/CML_k$
Thor Johnson and Robin Thomas

Theorem (Theorem 1)

$G$ & $H$ internally 4-connected graphs and $H \not\subseteq G$ and, if $H$ is a $\text{CPL}_k/\text{CML}_k/\text{CPB-W}_k$
Theorem (Theorem 1)

G & H internally 4-connected graphs and H $\not\subseteq$ G and, if H is a CPL$_k$/CML$_k$/CPB-W$_k$/CMB-W$_k$
Theorem (Theorem 1)

$G \& H$ internally 4-connected graphs and $H \not\leq G$ and, if $H$ is a $CPL_k/CML_k/CPB-W_k/CMB-W_k$, then $G$ has no $QPL_k$
Theorem (Theorem 1)

\(G \& H\) internally 4-connected graphs and \(H \nsubseteq G\) and, if \(H\) is a \(\text{CPL}_k/\text{CML}_k/\text{CPB}-W_k/\text{CMB}-W_k\), then \(G\) has no \(\text{QPL}_k/\text{QML}_{k+1}\)
Theorem (Theorem 1)

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Thor Johnson and Robin Thomas

Theorem (Theorem 1)

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Theorem (Theorem 1)

$G$ & $H$ internally 4-connected graphs and $H \not\cong G$ and, if $H$ is a $CPL_k/CML_k/CPB-W_k/CMB-W_k$, then $G$ has no $QPL_k/QML_{k+1}/QPB-W_k/QMB-W_k$-minor $\implies \exists H' \not\cong G$ where
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(i) $H'/e = H$ for some edge $e$; or
Theorem (Theorem 1)

$G \& H$ internally 4-connected graphs and $H \nsubseteq G$ and, if $H$ is a $\text{CPL}_k/\text{CML}_k/\text{CPB-W}_k/\text{CMB-W}_k$, then $G$ has no $\text{QPL}_k/\text{QML}_{k+1}/\text{QPB-W}_k/\text{QMB-W}_k$-minor $\implies \exists H' \nsubseteq G$ where

(i) $H'/e = H$ for some edge $e$; or

(ii) $E(H') = E(H) \cup \{e_1, e_2, \ldots, e_t\}$, where $(V(H'), E(H) \cup \{e_1, e_2, \ldots, e_i\})$ has at most one 4-fan for all $i \in \{1, 2, \ldots, t - 1\}$ and $(V(H'), E(H) \cup \{e_1, e_2, \ldots, e_t\})$ is internally 4-connected; or
Theorem (Theorem 1)

$G$ & $H$ internally 4-connected graphs and $H \not\subseteq G$ and, if $H$ is a $CPL_k/CML_k/CPB-W_k/CMB-W_k$, then $G$ has no $QPL_k/QML_{k+1}/QPB-W_k/QMB-W_k$-minor $\implies \exists H' \not\subseteq G$ where

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(iii) $H'$ is obtained from $H$ by a quadrangular, pentagonal, or hexagonal extension.
Theorem (Theorem 1)

\(G \& H\) internally 4-connected graphs and \(H \nsubseteq G\) and, if \(H\) is a \(\text{CPL}_k/\text{CML}_k/\text{CPB-W}_k/\text{CMB-W}_k\), then \(G\) has no \(\text{QPL}_k/\text{QML}_{k+1}/\text{QPB-W}_k/\text{QMB-W}_k\)-minor \(\implies \exists H' \nsubseteq G\) where

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\((V(H'), E(H) \cup \{e_1, e_2, \ldots, e_i\})\) has at most one 4-fan for all \(i \in \{1, 2, \ldots, t - 1\}\) and \((V(H'), E(H) \cup \{e_1, e_2, \ldots, e_i\})\) is internally 4-connected; or

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Theorem (Theorem 2)

$G$ & $H$ internally 4-connected graphs and $H \nsubseteq G$ and, if $H$ is a $CPL_k/CML_k/CPB-W_k/CMB-W_k$, then $G$ has no $QPL_k/QML_{k+1}/QPB-W_k/QMB-W_k$-minor.
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AND $H \nsubseteq K_{3,3}$, Cube, and $G \nsubseteq CPL,CML,CPB-W,CMB-W$, and for $V' \subseteq V(H)$ the set of degree-3 vertices in $H$, the graph $H[V']$ has circuits and trees as its components
Theorem (Theorem 2)

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AND $H \not\cong K_{3,3}$, $Cube$, and $G \not\cong CPL,CML,CPB-W,CMB-W$, and for $V' \subseteq V(H)$ the set of degree-3 vertices in $H$, the graph $H[V']$ has circuits and trees as its components $\implies \exists H' \not\cong G$ where
Theorem (Theorem 2)

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AND $H \not\cong K_{3,3}$, Cube, and $G \not\cong \text{CPL},\text{CML},\text{CPB-W},\text{CMB-W}$, and for $V' \subseteq V(H)$ the set of degree-3 vertices in $H$, the graph $H[V']$ has circuits and trees as its components $\implies \exists H' \not\cong G$ where

(i) $H'/e = H$ for some edge $e$; or

(ii) $H'\setminus e = H$ for some edge $e$; or

(iii) $H'$ is obtained from $H$ by a quadrangular, pentagonal, or hexagonal extension.
$K_n$
Internally 4-connected binary matroids

How can we get an internally 4-connected minor of this?
Internally 4-connected binary matroids

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Internally 4-connected binary matroids

Theorem

$M, N$ internally 4-connected binary matroids & $N \not\leq M \implies \exists M',$ internally 4-connected, where $N \not\leq M' \not\leq M$

&
Theorem

\[ M, N \text{ internally 4-connected binary matroids} \land N \not\preceq M \implies \exists M', \text{ internally 4-connected, where } N \preceq M' \not\preceq M \land |E(M)| - |E(M')| \leq 3 \text{ or, (up to duality),} \]
Internally 4-connected binary matroids

**Theorem**

$M, N$ internally 4-connected binary matroids & $N \leq M \nRightarrow \exists M',$ internally 4-connected, where $N \leq M' \leq M$

& $|E(M)| - |E(M')| \leq 3$ or, (up to duality),

(i) $|E(M)| - |E(M')| = 4$ &
Theorem

$M, N$ internally 4-connected binary matroids $\& N \lesssim M \implies \exists M', \text{ internally 4-connected, where } N \preceq M' \preceq M$

$\& |E(M)| - |E(M')| \leq 3 \text{ or, (up to duality), }$

(i) $|E(M)| - |E(M')| = 4 \&$

get $M'$ from $M$ by a central cocircuit deletion in a good augmented 4-wheel or a ladder-compression move; or . . .
Internally 4-connected binary matroids

**Theorem**

\( M, N \) internally 4-connected binary matroids \& \( N \not\preceq M \implies \exists M', \) internally 4-connected, where \( N \preceq M' \preceq M \)

\& \( |E(M)| - |E(M')| \leq 3 \) or, (up to duality),

(i)  

(ii) get \( M' \) from \( M \) by trimming a double fan, a bowtie ring, or a “ladder;” or . . .
Internally 4-connected binary matroids

Theorem

Let $M, N$ be internally 4-connected binary matroids such that $N \subsetneq M$. Then there exists $M'$, internally 4-connected, where $N \subseteq M' \subseteq M$ and $|E(M)| - |E(M')| \leq 3$ or, (up to duality),

(i) get $M'$ from $M$ by trimming a double fan, a bowtie ring, or a "ladder;" or . . .

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Internally 4-connected binary matroids

Theorem

\( M, N \) internally 4-connected binary matroids & \( N \preceq M \implies \exists M', \text{ internally 4-connected, where } N \preceq M' \preceq M \)

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Internally 4-connected binary matroids

(iii) get $M'$ from $M$ by an enhanced-ladder move or lobster move; or . . .
(iii) get $M'$ from $M$ by an enhanced-ladder move or lobster move; or \ldots
Internally 4-connected binary matroids

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Internally 4-connected binary matroids

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Internally 4-connected binary matroids

**Theorem**

\( M, N \) internally 4-connected binary matroids & \( N \preceq M \implies \exists M', \text{ internally 4-connected}, \text{ where } N \preceq M' \preceq M \)

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& $|E(M)| - |E(M')| \leq 3$ or, (up to duality),

(i)

(ii)

(iii)

or . . .
Internally 4-connected binary matroids

(iv) \((M, N)\) is (QML,CML) or \ldots
Internally 4-connected binary matroids

(iv) \((M, N)\) is (QML,CML) or \((T^*\text{MM},\text{TMM})\); or . . .
Internally 4-connected binary matroids

(iv) $(M, N)$ is $(QML,CML)$ or $((T^*MM)^*,(TMM)^*)$; or \ldots
Internally 4-connected binary matroids

(iv) \((M, N)\) is (QML, CML) or \(((T^*MM)^*,(TMM)^*)\); or \ldots\
(v) \((M, N)\) is one of 31 *interesting pairs* (A splitter theorem for internally 4-connected binary matroids: small matroids, arXiv).

*interesting pair* (n.) \(|E(M)| \leq 15\) and \(M\) and \(N\) are internally 4-connected, but no proper minor of \(M\) with a proper \(N\)-minor is internally 4-connected
Splitter Theorem for internally 4-connected binary matroids

Theorem

\( M, N \) internally 4-connected binary matroids & \( N \preceq M \implies \exists M', \text{ internally 4-connected, where } N \preceq M' \preceq M \) & |
\( E(M) \) − |\( E(M') \) | \( \leq 3 \) or \( M \) and \( N \) are Möbius ladders or Möbius matroids or comprise an interesting pair or, (up to duality), \( M' \) is obtained by one of these moves in \( M \):

(i)

(ii)

(iii)
Compare and contrast

Johnson & Thomas splitter theorem (Theorem 1):

(i) require, if $H$ has a certain structure, then $G$ avoids certain minor; and
(ii) have one unbounded move from $H$ to get an internally 4-connected intermediate graph; and
(iii) have four nice bounded ($\leq 4$) moves from $H$ to get an intermediate graph (might not be internally 4-connected); and
(iv) is useful for finding excluded minor results.

Chun, Mayhew, & Oxley splitter theorem

(i) give no restriction for $N$; and
(ii) have several highly-structured unbounded moves from $M$ to internally 4-connected intermediate matroid; and
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Splitter Theorem for internally 4-connected binary matroids

**Theorem**

\[ M, N \text{ internally 4-connected binary matroids } \land N \not\preceq M \implies \exists M', \text{ internally 4-connected, where } N \preceq M' \not\preceq M \land |E(M)| - |E(M')| \leq 3 \text{ or } M \text{ and } N \text{ are Möbius ladders or Möbius matroids or comprise an interesting pair or, (up to duality), } M' \text{ is obtained by one of these moves in } M: \]

(i) \[
\begin{array}{c}
\includegraphics[width=0.3\textwidth]{figure1}
\end{array}
\]

(ii) \[
\begin{array}{c}
\includegraphics[width=0.3\textwidth]{figure2}
\end{array}
\]

(iii) \[
\begin{array}{c}
\includegraphics[width=0.3\textwidth]{figure3}
\end{array}
\]
Cor: Splitter Theorem for internally 4-connected graphs

**Theorem**

$G, H$ internally 4-connected graphs & $H \preceq G \implies \exists G',$ internally 4-connected, where $H \preceq G' \preceq G$ & $|E(G)| - |E(G')| \leq 3$ or $G$ and $H$ are Möbius ladders or comprise one of 12 interesting pairs or, (up to duality), $G'$ is obtained by one of these moves in $G$:

(i)

(ii)

(iii)
Cor: back to internally 4-connected binary matroids

Theorem

\( M, N \) internally 4-connected binary matroids \& \( N \preceq M \implies \exists M', \text{ internally 4-connected, where } N \preceq M' \preceq M \& |E(M)| - |E(M')| \leq 6 \) or \( M \) and \( N \) are Möbius ladders or Möbius matroids or, (up to duality), \( M' \) is obtained by one of these moves in \( M \):

(i)

(ii)