On the Description of Free Multiplicative Convolution in Terms of Monotone Cumulants

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Abstract

Cumulants are important tools in the study of non-commutative probability. This essay discusses foundational results regarding relationships between various brands of cumulants. Furthermore, we investigate the monotone cumulants of products of freely independent variables by obtaining new computational results.

Contents

1 Introduction

Various distinct notions of independence have arisen as tools to describe the distributions of products of non-commutative random variables. For each notion of independence, there is an accompanying family of multilinear cumulant functionals designed to operate naturally on suitably-independent arguments. This essay investigates the monotone cumulants of products of freely independent non-commutative random variables.

We begin in Section 2 by reviewing the notion of free independence in a non-commutative probability space. Section 3 introduces the family of free cumulant functionals, which grant combinatorial insight into free independence. In Section 4, the Boolean and monotone cumulants are defined, with a brief discussion of the conversion between families of cumulants. Finally, these conversions are used to compute low-order formulas describing free multiplicative convolution in terms of monotone cumulants. The resulting formulas, as well as the general method used to compute them, are presented in Section 5.

2 Free Probability

2.1 The Framework of Non-Commutative Probability

The objects of study in free probability are non-commutative probability spaces, a generalization of classical probability spaces stripped of analytical structure.

Definition 2.1. A **non-commutative probability space** $(\mathcal{A}, \varphi)$ consists of a unital algebra $\mathcal{A}$ over $\mathbb{C}$ and a linear functional $\varphi : \mathcal{A} \to \mathbb{C}$ satisfying $\varphi(1_{\mathcal{A}}) = 1$. We refer to elements $a \in \mathcal{A}$ as **non-commutative random variables**, or simply as **random variables**.

While the literature [9, Lec. 1] contains numerous additional restrictions that can be imposed on non-commutative probability spaces for a richer theory, this minimal framework is already sufficient to develop various notions of non-commutative independence.

Definition 2.2. We may think of $\varphi(a)$ as the expectation of the random variable $a \in \mathcal{A}$, analogous to the mean of a probability density function. Furthering this parallel, we call the sequence of scalars $(\varphi(a^n))_{n=1}^{\infty}$ the **moments of** $\boldsymbol{a}$.

A central goal in non-commutative probability theory is a description of the behaviour of φ on products of random variables. In classical probability, this question is addressed by a notion of *independence* between families of random variables. In practice, independence serves as a computational rule to compute the moments of products from existing knowledge of each individual variable's moments.

Intuitively, one might expect independence to allow the moments of monomials formed by the variables $a, b \in \mathcal{A}$ to be expressed in terms of values from $(\varphi(a^n))_{n=1}^{\infty}$ and $(\varphi(b^n))_{n=1}^{\infty}$. For example, classical probability suggests that one should have $\varphi(a^n b^m) = \varphi(a^n)\varphi(b^m)$ when a and b are independent (in fact, this will indeed hold of freely independent variables). However, since $\mathcal{A}$ is not necessarily commutative, this rule alone would be insufficient to address all such monomials. For instance, it is entirely possible to have random variables $a, b \in \mathcal{A}$ with $\varphi(a^{n_1} b^{m_1} a^{n_2} b^{m_2}) \neq \varphi(a^{n_1+n_2} b^{m_1+m_2})$. The next subsection introduces the notion of free independence, which serves as a more encompassing (albeit more complicated) computational rule to deal with such products.

2.2 Free Independence

Definition 2.3. Let $(\mathcal{A}, \varphi)$ be a non-commutative probability space and Λ an index set. For each $\lambda \in \Lambda$, let $\mathcal{A}_\lambda \subseteq \mathcal{A}$ be a unital subalgebra. We say $(\mathcal{A}_\lambda)_{\lambda\in\Lambda}$ are **freely indendent** if

$$\varphi(a_1 a_2 \cdots a_n) = 0$$

whenever $n \geq 2$ and the following hold:

- $\varphi(a_i) = 0$ for $1 \leq i \leq n$
- $a_i \in \mathcal{A}_{\lambda(i)}$ with $\lambda(i) \in \Lambda$ for $1 \leq i \leq n$
- $\lambda(i) \neq \lambda(i+1)$ for $1 \leq i < n$ (ie. adjacent elements are from different subalgebras)

Furthermore, suppose $(a_\lambda)_{\lambda\in\Lambda}$ is a family in $\mathcal{A}$. For each $\lambda \in \Lambda$, let $\mathcal{A}_\lambda$ be the unital subalgebra generated by a_λ. We say $(a_\lambda)_{\lambda\in\Lambda}$ are **freely independent** if $(\mathcal{A}_\lambda)_{\lambda\in\Lambda}$ are freely independent.

Remark 2.4. The unital subalgebra generated by $a \in \mathcal{A}$ is given explicitly by $\{P(a) : P \in \mathbb{C}[x]\}$ (univariate polynomials evaluated at a).

Although this definition may not seem entirely natural at first, it does provide a mechanism to compute moments of products. For example, suppose $a, b \in \mathcal{A}$ are freely independent. By defining

$$a^\circ := a - \varphi(a)1, \quad b^\circ := b - \varphi(b)1,$$

we obtain $\varphi(a^\circ) = 0 = \varphi(b^\circ)$. This technique is known as *centering.* The definition of free independence thus yields

$$\begin{aligned}
\varphi(ab) &= \varphi((a^\circ + \varphi(a)1)(b^\circ + \varphi(b)1)) \\
&= \varphi(a^\circ b^\circ) + \varphi(a)\varphi(b^\circ) + \varphi(b)\varphi(a^\circ) + \varphi(a)\varphi(b) \\
&= \varphi(a^\circ b^\circ) + \varphi(a)\varphi(b) = \varphi(a)\varphi(b)
\end{aligned}$$

With additional effort, one can use the same technique to arrive at the formula

$$\varphi(abab) = \varphi(a^2)\varphi(b)^2 + \varphi(a)^2\varphi(b^2) - \varphi(a)^2\varphi(b)^2$$

Indeed, the effort required to compute moments of ab in this way grows rapidly; the analogous formulas for $\varphi((ab)^3)$, $\varphi((ab)^4)$, and $\varphi((ab)^5)$ contain 12, 55, and 273 terms respectively. This apparent complexity motivates a combinatorial study of the resulting moment formulas.

3 Free Cumulants

The free cumulants are a family multilinear functionals introduced by Speicher [11] which allow free independence to be studied from a combinatorial viewpoint. Similar to the cumulants of classical probability, free cumulants contain the same information as moments; sufficient knowledge of one allows the other to be uniquely determined. Whereas classical cumulants have a combinatorial interpretation relating to *all* partitions of a finite set, the free cumulants depend only on *non-crossing* partitions.

Competing brands of cumulants, including the Boolean and monotone cumulants, have been introduced to study various notions of independence. Despite their nuances, all of the aforementioned brands of cumulants are intimately tied to the lattices of non-crossing partitions.

3.1 Non-Crossing Partitions

Definition 3.1. S be a totally-ordered set. Furthermore, suppose π is a partition of S. We call each $B \in \pi$ a **block** of π. We say π is **crossing** if there exist distinct blocks $B_1, B_2 \in \pi$ with $p_1, p_2 \in B_1$ and $q_1, q_2 \in B_2$ such that $p_1 < q_1 < p_2 < q_2$. Otherwise, we say π is **non-crossing**. We denote the set of all non-crossing partitions of S by $\boldsymbol{NC(S)}$, and define $\boldsymbol{NC(n)} := NC(\{1, 2, \cdots, n\}), \forall n \in \mathbb{N}$.

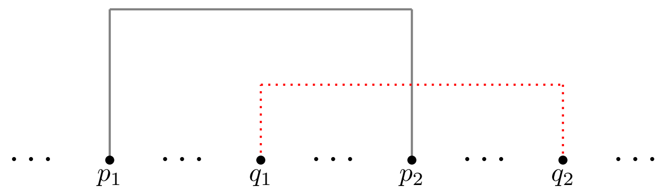

Figure 1: A crossing partition.

The non-crossing partitions were introduced by Kreweras [8] in 1972, and have since played important roles in various branches of algebraic combinatorics.

A common method of depicting non-crossing partitions graphically is by joining all members of the same block with a line above the horizontal. The non-crossing partition $\{\{1, 5, 6\}, \{2, 4\}, \{3\}, \{7\}\} \in NC(7)$ is depicted below.

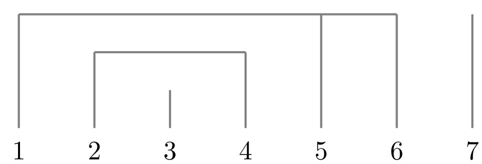

Figure 2: An element of $NC(7)$.

It is easily seen that all partitions of at most 3 elements are non-crossing, and that $\{\{1, 3\}, \{2, 4\}\}$ is the unique crossing partition of four elements. Simple verification yields $|NC(1)| = 1$, $|NC(2)| = 2$, $|NC(3)| = 5$, and $|NC(4)| = 14$. Below is a diagram of the 14 elements of $NC(4)$, organized into rows by number of blocks.

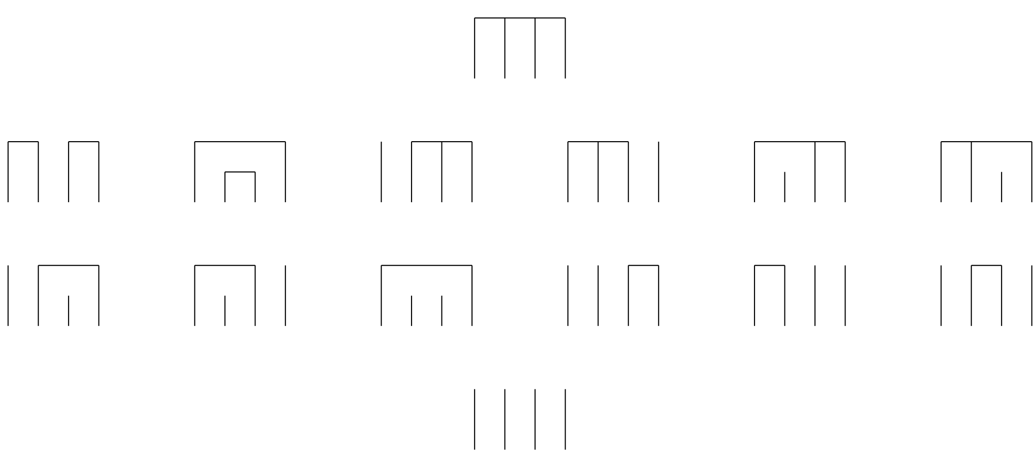

Figure 3: $NC(4)$

It is a standard combinatorial exercise to show that $NC(n)$ is counted by the n-th Catalan number.

$$|NC(n)| = C_n = \frac{1}{n+1}\binom{2n}{n}, \quad \forall n \in \mathbb{N}$$

Furthermore, it is known that the number of elements in each row of the corresponding diagram for $NC(n)$ is captured by the Narayana numbers [9, Cor. 9.13].

$$|\{\pi \in NC(n) : |\pi| = k\}| = N(n,k) = \frac{1}{n}\binom{n}{k}\binom{n}{k-1}, \quad \forall n \in \mathbb{N},\, 1 \leq k \leq n$$

Since $N(n,k) = N(n, n+1-k)$, the rows of $NC(n)$ exhibit a form of vertical symmetry. This observation can be further explored by first introducing a partial ordering on each $NC(n)$.

Definition 3.2. A particularly natural partial ordering on each set $NC(n)$ is the **reverese-refinement order**. Let $n \in \mathbb{N}$ and $\pi, \sigma \in NC(n)$. We say $\pi \leq \sigma$ to mean:

$$\text{For each block } V \in \pi, \text{ there exists } W \in \sigma \text{ such that } V \subseteq W.$$

Equivalently, $\pi \leq \sigma$ precisely when each block of σ is a union of blocks of π.

Remark 3.3. The reverse-refinement order is known to induce a lattice structure on $NC(n)$ with notably elegant isomorphism results concerning subintervals [9, Lec. 9].

Definition 3.4. For each $n \in \mathbb{N}$, we define

$$\mathbf{1_n} := \{\{1, 2, \cdots, n\}\},$$

the unique maximal element of $NC(n)$ with respect to the reverse-refinement order.

Definition 3.5. An important transformation of non-crossing partitions is the **Kreweras complement Kr** : $NC(n) \to NC(n)$, which can be defined as follows: Let $n \in \mathbb{N}$ and $\pi \in NC(n)$. For $1 \leq i \leq n$, we denote $\bar{i} := i + 1/2$. $\mathrm{Kr}(\pi)$ is defined to be the largest $\sigma \in NC(\{\bar{1}, \cdots, \bar{n}\})$ (by reverse-refinement order) satisfying $\pi \cup \sigma \in NC(\{1, \bar{1}, \cdots, n, \bar{n}\})$.

For example, suppose $\pi = \{\{1,5,6\},\{2,4\},\{3\},\{7\}\} \in NC(7)$. We obtain $\mathrm{Kr}(\pi) = \{\{1,4\},\{2,3\},\{5\},\{6,7\}\} \in NC(7)$, as captured by the illustration below.

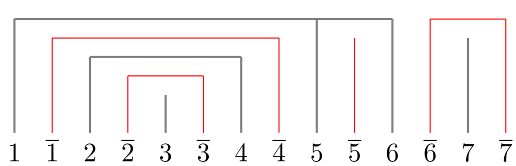

Figure 4: $\pi, \mathrm{Kr}(\pi) \in NC(7)$

The Kreweras complementation map exhibits many interesting properties. In particular, it can be shown that Kr is a lattice anti-isomorphism. That is, Kr is a bijection satisfying

$$\pi \leq \sigma \iff \mathrm{Kr}(\sigma) \leq \mathrm{Kr}(\pi), \quad \forall \pi, \sigma \in NC(n)$$

In effect, the Kreweras complement serves as a method of flipping the lattice $NC(n)$ upside-down. Consequently, we obtain

$$|\pi| + |\mathrm{Kr}(\pi)| = n + 1, \quad \forall n \in \mathbb{N},\, \pi \in NC(n) \tag{3.1}$$

Remark 3.6. The Kreweras complement has a practical interpretation in terms of the symmetric group $\mathcal{S}_n$. For each non-empty $B = \{i_1, \cdots, i_k\} \subseteq \{1, \cdots, n\}$ with $i_1 < i_2 < \cdots < i_k$, we denote the cyclic permutation $\phi_B := (i_1, \cdots, i_k) \in \mathcal{S}_n$. The embedding $\Phi : NC(n) \to \mathcal{S}_n$ defined by

$$\Phi(\pi) := \prod_{B \in \pi} \phi_B \tag{3.2}$$

provides the following method for computing the value of $\mathrm{Kr}(\pi)$ algebraically [3, Sec. 1.4.2].

$$\Phi(\mathrm{Kr}(\pi)) = \Phi(\pi)^{-1}\Phi(1_n), \quad \forall n \in \mathbb{N}, \pi \in NC(n) \tag{3.3}$$

3.2 Important Families of Non-Crossing Partitions

Non-crossing partitions of particular forms play important roles in the conversion between moments and varying brands of cumulants. Two notable families of non-crossing partitions are given below, with examples shown in Figure 5.

Definition 3.7. Let $n \in \mathbb{N}$ and $\pi \in NC(n)$. If every block $B \in \pi$ is of the form $[i, j] \cap \mathbb{N}$ for some $1 \leq i \leq j \leq n$, then π is said to be an **interval partition**. We define

$$NC_{\mathrm{int}}(n) := \{\pi \in NC(n) : \pi \text{ is an interval partition } \}, \quad \forall n \in \mathbb{N}$$

It is not difficult to see that $|NC_{\mathrm{int}}(n)| = 2^{n-1}$ for all $n \in \mathbb{N}$.

Definition 3.8. Let $n \in \mathbb{N}$ and $\pi \in NC(n)$. If there exists a block $B \in \pi$ with $1, n \in B$, then π is said to be **irreducible**. Note that $1_1 = \{\{1\}\} \in NC(1)$ is irreducible. We define

$$NC_{\mathrm{irr}}(n) := \{\pi \in NC(n) : \pi \text{ is irreducible}\}, \quad \forall n \in \mathbb{N}$$

It is not difficult to see that $|NC_{\mathrm{irr}}(n)| = C_{n-1}$ for all $n \in \mathbb{N}$.

Figure 5: Examples of certain varieties of non-crossing partitions in $NC(6)$.

3.3 The Free Cumulant Functionals

Let $(\mathcal{A}, \varphi)$ be a non-commutative probability space. Before defining the family of free cumulant functionals, we introduce some useful notation. Let $n \in \mathbb{N}$ and $B = \{i_1, i_2, \cdots, i_{|B|}\}$ be a non-empty subset of $\{1, 2, \cdots, n\}$ with $i_1 < i_2 < \cdots < i_{|B|}$. The following is notation for the restriction of an n-tuple to the indices of B.

$$(a_1, a_2, \cdots, a_n) \mid B := (a_{i_1}, a_{i_2}, \cdots, a_{i_{|B|}}), \quad \forall (a_1, a_2, \cdots, a_n) \in \mathcal{A}^n \tag{3.4}$$

Now let $(f_n : \mathcal{A}^n \to \mathbb{C})_{n=1}^{\infty}$ be a family of multilinear functionals. We extend this family to a new collection of multilinear functionals $(f_\pi : \mathcal{A}^n \to \mathbb{C})_{n \in \mathbb{N}, \pi \in NC(n)}$ indexed by non-crossing partitions as follows:

$$f_\pi(a_1, \cdots, a_n) := \prod_{B \in \pi} f_{|B|}\big((a_1, \cdots, a_n) \mid B\big), \quad \forall n \in \mathbb{N}, \pi \in NC(n), (a_1, \cdots, a_n) \in \mathcal{A}^n \tag{3.5}$$

From this definition, it is immediately seen that $f_{1_n} = f_n$ for all $n \in \mathbb{N}$. For a non-crossing partition $\pi \in \bigcup_{n=1}^{\infty} NC(n)$, f_π can be thought of as distributing its arguments according to the block structure of π. As a concrete example, take $\pi = \{\{1,3,6\},\{2\},\{4,5\}\} \in NC(6)$. The following holds.

$$f_\pi(a_1, \cdots, a_6) = f_3(a_1, a_3, a_6) \cdot f_1(a_2) \cdot f_2(a_4, a_5), \quad \forall (a_1, \cdots, a_6) \in \mathcal{A}^6$$

Definition 3.9. For each $n \in \mathbb{N}$, there exists a corresponding multilinear functional $\kappa_n : \mathcal{A}^n \to \mathbb{C}$ named the ***n*-th free cumulant functional**. The family $(\kappa_n)_{n \in \mathbb{N}}$ is uniquely determined by the following recursive moment-cumulant formula.

$$\varphi(a_1 \cdots a_n) = \sum_{\pi \in NC(n)} \kappa_\pi(a_1, \cdots, a_n), \quad \forall n \in \mathbb{N},\, (a_1, \cdots, a_n) \in \mathcal{A}^n \tag{3.6}$$

It is readily seen that $\varphi(a) = \kappa_1(a)$ for all $a \in \mathcal{A}$, hence $\kappa_1 = \varphi$. By rearranging the above moment-cumulant formula, the values of κ_n for $n \geq 2$ can be obtained from known values of the free cumulant functionals of strictly lower arity.

$$\kappa_n(a_1, \cdots, a_n) = \varphi(a_1 \cdots a_n) - \sum_{\substack{\pi \in NC(n) \\ \pi \neq 1_n}} \kappa_\pi(a_1, \cdots, a_n), \quad \forall n \geq 2,\, (a_1, \cdots, a_n) \in \mathcal{A}^n \tag{3.7}$$

Using this recursive formula, the values of κ_n for small n are computed explicitly below.

$$\kappa_2(a_1, a_2) = \varphi(a_1 a_2) - \kappa_1(a_1)\kappa_1(a_2) = \varphi(a_1 a_2) - \varphi(a_1)\varphi(a_2)$$

$$\begin{aligned} \kappa_3(a_1, a_2, a_3) = {} & \varphi(a_1 a_2 a_3) - \kappa_1(a_1)\kappa_2(a_2, a_3) - \kappa_1(a_2)\kappa_2(a_1, a_3) \\ & -\kappa_1(a_3)\kappa_2(a_1, a_2) - \kappa_1(a_1)\kappa_1(a_2)\kappa_1(a_3) \\ = {} & \varphi(a_1 a_2 a_3) - \varphi(a_1)\varphi(a_2 a_3) - \varphi(a_2)\varphi(a_1 a_3) \\ & -\varphi(a_3)\varphi(a_1 a_2) + 2\varphi(a_1)\varphi(a_2)\varphi(a_3) \end{aligned}$$

Restricting our attention to the case $a_1 = a_2 = \cdots = a_n$, we define the following notation.

$$\kappa_n(a) := \kappa_n(a, \cdots, a), \quad \kappa_\pi(a) := \kappa_\pi(a, \cdots, a) = \prod_{B \in \pi} \kappa_{|B|}(a), \quad \forall a \in \mathcal{A}, n \in \mathbb{N}, \pi \in NC(n) \tag{3.8}$$

Definition 3.10. Fix $a \in \mathcal{A}$. The sequence of scalars $(\kappa_n(a))_{n=1}^{\infty}$ is referred to as the **free cumulants of *a***. By Equation (3.7), a formula for $\kappa_n(a)$ with $n \geq 2$ is given by

$$\kappa_n(a) = \varphi(a^n) - \sum_{\substack{\pi \in NC(n) \\ \pi \neq 1_n}} \prod_{B \in \pi} \kappa_{|B|}(a) \tag{3.9}$$

Hence, for small values of n, we obtain

$$\kappa_1(a) = \varphi(a), \quad \kappa_2(a) = \varphi(a^2) - \varphi(a)^2$$

$$\kappa_3(a) = \varphi(a^3) - 3\varphi(a)\varphi(a^2) + 2\varphi(a)^3$$

Remark 3.11. It is seen that the map $a \mapsto \kappa_n(a)$ is a polynomial in the moments of a. Combining this observation with the moment-cumulant formula Equation (3.6), we remark that the moments of a random variable $a \in \mathcal{A}$ can be obtained from its free cumulants and vice-versa. That is, the moments and free cumulants of a contain the same information.

3.4 Connection to Free Independence

We return to the goal of describing the moments of products of freely independent random variables using existing knowledge of each term's individual moments. By Remark 3.11, it suffices to describe the *free cumulants* of such products using existing knowledge of each term's individual *free cumulants*.

Definition 3.12. Let $(\mathcal{A}, \varphi)$ be a non-commutative probability space, and $a, b \in \mathcal{A}$ be freely independent. For the purposes of this essay, we will refer to to map $(a, b) \mapsto a + b$ as **free convolution**. Similarly, we refer to the map $(a, b) \mapsto ab$ as **free multiplicative convolution**.

A fundamental property of the free cumulant functionals is known as the *vanishing of mixed cumulants*. This property, recorded in Theorem 3.13, leads to concrete descriptions of free convolution and free multiplicative convolution in terms of free cumulants.

Theorem 3.13 (Vanishing of Mixed Cumulants).
Let Λ be an index set, and $(\mathcal{A}_\lambda)_{\lambda\in\Lambda}$ be a family of unital subalgebras of $\mathcal{A}$. The following are equivalent:

(i) $(A_\lambda)_{\lambda\in\Lambda}$ are freely independent.

(ii) For all $n \geq 2$, we have $\kappa_n(a_1, \cdots, a_n) = 0$ whenever the following hold:

- $a_i \in \mathcal{A}_{\lambda(i)}$ with $\lambda(i) \in \Lambda$ for $1 \leq i \leq n$.
- There exists $1 \leq j, k \leq n$ with $\lambda(j) \neq \lambda(k)$.

For a detailed proof, see [9, Thm. 11.16]. As a consequence of the implication (i) $\rightarrow$ (ii), we obtain a simple description of free convolution in terms of free cumulants.

Corollary 3.14. Let $n \in \mathbb{N}$ and $a, b \in \mathcal{A}$ be freely independent. The following holds.

$$\kappa_n(a + b) = \kappa_n(a) + \kappa_n(b) \tag{3.10}$$

Proof. Multilinearity of κ_n and the vanishing of mixed cumulants yield:

$$\begin{aligned}\kappa_n(a + b) = \kappa_n(a + b, \cdots, a + b) &= \sum_{S\in\{a,b\}^n} \kappa_n(S) \\ &= \kappa_n(a, \cdots, a) + \kappa_n(b, \cdots, b) = \kappa_n(a) + \kappa_n(b)\end{aligned}$$

□

With additional effort, similar reasoning can be used to show that free multiplicative convolution is also readily expressed in terms of free cumulants.

Theorem 3.15. Let $n \in \mathbb{N}$ and $a, b \in \mathcal{A}$ be freely independent. The following holds.

$$\kappa_n(ab) = \sum_{\pi\in NC(n)} \kappa_\pi(a)\kappa_{\mathrm{Kr}(\pi)}(b) \tag{3.11}$$

This formula was first shown to hold by Nica and Speicher [10], who have since simplified the proof by capitalizing on the vanishing of mixed cumulants [9, Lec. 14].

4 Alternative Families of Cumulants

Theorem 3.13 and its consequences indicate a fundamental connection between the free cumulants and free independence. Analogously, several related families of cumulant functionals have emerged to address alternative notions of non-commutative independence [7].

4.1 Boolean and Monotone Cumulants

The Boolean cumulant functionals, introduced by Speicher and Woroudi [12], are designed to describe the notion of *Boolean independence*. Unlike free cumulants which are defined in terms of *all* non-crossing partitions, the Boolean cumulants are defined in terms of only interval partitions.

Definition 4.1. For each $n \in \mathbb{N}$, there exists a corresponding multilinear functional $\beta_n : \mathcal{A}^n \to \mathbb{C}$ named the ***n*-th Boolean cumulant functional**. The family $(\beta_n)_{n\in\mathbb{N}}$ is uniquely determined by the following recursive moment-cumulant formula.

$$\varphi(a_1 \cdots a_n) = \sum_{\pi \in NC_{\text{int}}(n)} \beta_\pi(a_1, \cdots, a_n), \quad \forall n \in \mathbb{N},\, (a_1, \cdots, a_n) \in \mathcal{A}^n \tag{4.1}$$

Of particular interest is that an analogue of Theorem (3.15) holds for Boolean cumulants [2].

$$\beta_n(ab) = \sum_{\pi \in NC(n)} \beta_\pi(a)\beta_{\mathrm{Kr}(\pi)}(b), \quad \forall n \in \mathbb{N},\, \forall a, b \in \mathcal{A} \text{ freely independent} \tag{4.2}$$

This result can be considered surprising, as Boolean cumulants are designed to address Boolean independence rather than free independence.

In a similar fashion to the free and Boolean cumulants, monotone cumulants were introduced by Hasebe [6] to describe the competing notion of *monotone independence*. The monotone cumulant functionals rely on the *tree factorial* function, which is defined in terms of rooted forests in [1, Sec. 3]. An equivalent definition, in terms of only non-crossing partitions, is given below.

Definition 4.2. Let $n \in \mathbb{N}$ and $\pi \in NC(n)$. For blocks $B_1, B_2 \in \pi$, we say B_1 is a **child** of B_2 when the following holds.

$$\min(B_2) < \min(B_1) \text{ and } \max(B_1) < \max(B_2) \tag{4.3}$$

The **tree factorial** of π, denoted $\boldsymbol{\tau(\pi)!}$, can be defined by

$$\tau(\pi)! := \prod_{B \in \pi} \big(1 + |\{B' \in \pi : B' \text{ is a child of } B\}|\big) \tag{4.4}$$

For example, when $\pi = \{\{1,5,7\},\{2,4\},\{3\},\{6\}\} \in NC(7)$, the block $\{1,5,7\}$ has precisely 3 children. Hence, it is seen that $\tau(\pi)! = 4 \cdot 2 \cdot 1 \cdot 1 = 8$. Note that $\tau(1_n)! = 1$ for all $n \in \mathbb{N}$.

Definition 4.3. For each $n \in \mathbb{N}$, there exists a corresponding multilinear functional $\rho_n : \mathcal{A}^n \to \mathbb{C}$ named the ***n*-th monotone cumulant functional**. The family $(\rho_n)_{n\in\mathbb{N}}$ is uniquely determined by the following recursive moment-cumulant formula.

$$\varphi(a_1 \cdots a_n) = \sum_{\pi \in NC(n)} \frac{1}{\tau(\pi)!} \rho_\pi(a_1, \cdots, a_n), \quad \forall n \in \mathbb{N},\, (a_1, \cdots, a_n) \in \mathcal{A}^n \tag{4.5}$$

As before with free cumulants, we define the following notation:

$$\beta_n(a) := \beta_n(a, \cdots, a), \quad \beta_\pi(a) := \beta_\pi(a, \cdots, a) = \prod_{B \in \pi} \beta_{|B|}(a), \quad \forall a \in \mathcal{A}, n \in \mathbb{N}, \pi \in NC(n) \tag{4.6}$$

$$\rho_n(a) := \rho_n(a, \cdots, a), \quad \rho_\pi(a) := \rho_\pi(a, \cdots, a) = \prod_{B \in \pi} \rho_{|B|}(a), \quad \forall a \in \mathcal{A}, n \in \mathbb{N}, \pi \in NC(n) \tag{4.7}$$

Definition 4.4. Fix $a \in \mathcal{A}$. The values $(\beta_n(a))_{n=1}^\infty$ are referred to as the **Boolean cumulants of *a***. Analogously, the values $(\rho_n(a))_{n=1}^\infty$ are referred to as the **monotone cumulants of *a***.

4.2 Transitions Between Cumulants

As seen in the previous sections, each brand of cumulant is defined to satisfy an invertible moment-cumulant formula. Therefore, one should expect the existence of cumulant-cumulant formulas which enable conversions between different brands of cumulants.

Indeed, for fixed $n \in \mathbb{N}$, the formula describing conversion from one brand of cumulant to another can be computed algorithmically. For example, $\beta_n(a)$ can be expressed as a polynomial in the monotone cumulants of $a \in \mathcal{A}$ by performing the following two steps.

1. Analogously to Equation (3.7), invert the upper-triangular system in Equation (4.1) to solve for $\beta_n(a)$ as a polynomial in the moments of a.
2. As suggested by the moment-cumulant relation in Equation (4.5), perform the substitution

$$\varphi(a^k) \mapsto \sum_{\pi \in NC(k)} \frac{1}{\tau(\pi)!} \rho_\pi(a), \quad 1 \leq k \leq n$$

to each term of the expression from Step 1. The result is a formula for $\beta_n(a)$ that is polynomial in the monotone cumulants of a.

Fortunately, laborious computations of this nature can often be avoided. Several general formulas describing transitions between certain brands of cumulants have been shown to hold. Two particularly relevant examples are given in [1, Thm. 1.1].

Theorem 4.5.

$$\beta_n(a) = \sum_{\pi \in NC_{\text{irr}}(n)} \frac{1}{\tau(\pi)!} \rho_\pi(a), \quad \forall n \in \mathbb{N},\, a \in \mathcal{A} \tag{4.8}$$

$$\kappa_n(a) = \sum_{\pi \in NC_{\text{irr}}(n)} \frac{(-1)^{|\pi|-1}}{\tau(\pi)!} \rho_\pi(a), \quad \forall n \in \mathbb{N},\, a \in \mathcal{A} \tag{4.9}$$

5 Monotone Cumulants of Freely Independent Products

Due to the surprising effectiveness of Boolean cumulants to operate on products of freely independent random variables (evidenced by Equation (4.2) and presented more broadly in [5]) , we aim to determine if a similar phenomenon occurs for monotone cumulants.

5.1 The Setup

Let $(\mathcal{A}, \varphi)$ a non-commutative probability space, and $a, b \in \mathcal{A}$ be freely independent. For each $n \in \mathbb{N}$, we wish to express $\rho_n(ab)$ in terms of the monotone cumulants of a and b separately. As seen in Equations (3.11) and (4.2), the following formulas are known to hold:

$$\kappa_n(ab) = \sum_{\pi \in NC(n)} \kappa_\pi(a)\kappa_{\text{Kr}(\pi)}(b), \quad \forall n \in \mathbb{N}$$

$$\beta_n(ab) = \sum_{\pi \in NC(n)} \beta_\pi(a)\beta_{\text{Kr}(\pi)}(b), \quad \forall n \in \mathbb{N}$$

Wishful thinking might lead one to ponder

$$\rho_n(ab) \stackrel{?}{=} \sum_{\pi \in NC(n)} \rho_\pi(a)\rho_{\text{Kr}(\pi)}(b), \quad \forall n \in \mathbb{N}$$

At first glance, there is little concrete reason to believe that the same formula will hold (even approximately) for monotone cumulants. Despite this, we were able to verify that this equality does indeed hold for $n \leq 4$ and begins to fail at $n = 5$ with a difference of only one term. The process used to obtain a true formula for $\rho_n(ab)$ occurs in three steps:

1. Write $\rho_n(ab)$ in terms of the free cumulants of ab. A general formula for writing montone cumulants in terms of free cumulants is presented in [4, Thm. 5.6]. However, this result was not directly used in our computations. Instead, low-order formulas were computed as needed.

2. Use Theorem 3.15 to separate the free cumulants of ab. That is, perform the substitutions

$$\kappa_k(ab) \mapsto \sum_{\pi \in NC(k)} \kappa_\pi(a)\kappa_{\mathrm{Kr}(\pi)}(b), \quad 1 \leq k \leq n$$

   in each term of the resulting expression.

3. Finally, convert free cumulants back into monotone cumulants by performing the substitutions

$$\kappa_k(x) \mapsto \sum_{\pi \in NC_{\mathrm{irr}}(k)} \frac{(-1)^{|\pi|-1}}{\tau(\pi)!}\rho_\pi(x), \quad 1 \leq k \leq n,\, x \in \{a, b\}$$

   as justified by Equation (4.9).

Remark 5.1. An alternative procedure can be performed by instead converting to Boolean cumulants in Step 1 using Equation (4.8). Then, Equation (4.2) can be used to separate the Boolean cumulants of ab. To write Boolean cumulants back in terms of monotone cumulants, one can either compute low-order formulas as needed or use the general transition formulas presented in [4].

The result is an expression for $\rho_n(ab)$ that is polynomial in the monotone cumulants of a and the monotone cumulants of b. The precise value of this polynomial was computed for $n \leq 9$ and compared term by term with the expression $\sum_{\pi \in NC(n)} \rho_\pi(a)\rho_{\mathrm{Kr}(\pi)}(b)$.

5.2 Computational Results

Remark 5.2. For the sake of brevity, the formulas below for $n \geq 5$ are presented with the assumption $\rho_1(a) = \rho_1(b) = 1$. To recover the general formulas, apply the following transformation to each monomial.

$$\left(\prod_{i=2}^{n} \rho_i(a)^{j_i}\right)\left(\prod_{i=2}^{n} \rho_i(b)^{k_i}\right) \mapsto \left(\prod_{i=1}^{n} \rho_i(a)^{j_i}\right)\left(\prod_{i=1}^{n} \rho_i(b)^{k_i}\right)$$

Where we define

$$k_1 := n - \sum_{i=2}^{n} i \cdot k_i, \quad j_1 := n - \sum_{i=2}^{n} i \cdot j_i$$

E.g. when $n = 6$, we have $\rho_2(a)^2\rho_2(b) \mapsto \rho_1(a)^2\rho_2(a)^2\rho_1(b)^4\rho_2(b)$. Note that the "subscript sum" of each monomial after substitution is equal to $2n$, divided evenly between "a terms" and "b terms".

What follow are the computed formulas for $n \leq 7$. See the Appendix for $n = 8, 9$.

$$\rho_n(ab) - \sum_{\pi\in NC(n)} \rho_\pi(a)\rho_{\mathrm{Kr}(\pi)}(b) = 0, \quad n = 1,2,3,4$$

$$\rho_5(ab) - \sum_{\pi\in NC(5)} \rho_\pi(a)\rho_{\mathrm{Kr}(\pi)}(b) = -\frac{1}{12}\rho_2(a)\rho_2(b)$$

$$\rho_6(ab) - \sum_{\pi\in NC(6)} \rho_\pi(a)\rho_{\mathrm{Kr}(\pi)}(b) = -\frac{1}{4}\rho_2(a)^2\rho_2(b) - \frac{1}{4}\rho_2(a)\rho_2(b)^2 - \frac{1}{3}\rho_2(a)\rho_3(b) - \frac{1}{3}\rho_3(a)\rho_2(b)$$

$$\begin{aligned}&\rho_7(ab) - \sum_{\pi\in NC(7)} \rho_\pi(a)\rho_{\mathrm{Kr}(\pi)}(b) = \\ &- \rho_3(a)\rho_3(b) - \frac{4}{3}\rho_2(a)^2\rho_2(b)^2 - \frac{1}{6}\rho_2(a)^3\rho_2(b) - \frac{1}{6}\rho_2(a)\rho_2(b)^3 - \frac{3}{4}\rho_2(a)\rho_4(b) - \frac{3}{4}\rho_4(a)\rho_2(b) \\ &- \frac{19}{12}\rho_2(a)^2\rho_3(b) - \frac{19}{12}\rho_3(a)\rho_2(b)^2 - \frac{17}{12}\rho_2(a)\rho_3(a)\rho_2(b) - \frac{17}{12}\rho_2(a)\rho_2(b)\rho_3(b) + \frac{7}{180}\rho_2(a)\rho_2(b)\end{aligned}$$

5.3 Observations

We conclude with some modest observations regarding the computed formulas. We refer to the terms that appear on the right-hand side of each equation as "stray terms".

Observation 5.3. An immediate observation is that the the stray terms exhibit the following symmetry. For all monomials of the form

$$\left(\prod_{i=1}^{n} \rho_i(a)^{j_i}\right)\left(\prod_{i=1}^{n} \rho_i(b)^{k_i}\right)$$

there is a corresponding "mirrored" monomial

$$\left(\prod_{i=1}^{n} \rho_i(a)^{k_i}\right)\left(\prod_{i=1}^{n} \rho_i(b)^{j_i}\right)$$

with the same rational coefficient. In this sense, the stray terms come in pairs (except in the case when $j_i = k_i$ for $1 \le i \le n$).

Observation 5.4. By (3.1), it is seen that all terms in the expression

$$\sum_{\pi\in NC(n)} \rho_\pi(a)\rho_{\mathrm{Kr}(\pi)}(b)$$

have degree $n+1$ (where the monotone cumulants of a and b are viewed as indeterminants). However, none of the computed stray terms possess this degree. The degrees of stray terms, after performing the transformation indicated in Remark 5.2, are recorded in the following table.

Degree	n + 3	n + 5	n + 7	Total
n = 5	1	0	0	1
n = 6	4	0	0	4
n = 7	10	1	0	11
n = 8	20	4	0	24
n = 9	41	10	1	52

Hence it seems reasonable to conjecture that the analogous formula for $n = 10$ will contain 4 stray terms of degree 17, 20 of degree 15, and the rest having degree 13. Similarly, we expect the analogous formula for $n = 11$ to contain 1 stray term of degree 20, 10 of degree 18, 41 of degree 16, and the rest having degree 14.

For now, we leave the complete description of free multiplicative convolution in terms of monotone cumulants as an open problem.

Appendix

Below are the computed formulas describing free multiplicative convolution in terms of monotone cumulants for $n = 8, 9$ under the assumption $\rho_1(a) = \rho_1(b) = 1$. See Remark 5.2 for the procedure to recover the general formulas.

$$
\begin{aligned}
&\rho_8(ab) - \sum_{\pi \in NC(8)} \rho_\pi(a)\rho_{\mathrm{Kr}(\pi)}(b) = \\
&- 2\rho_2(a)^3\rho_2(b)^2 - 2\rho_2(a)^2\rho_2(b)^3 - \frac{8}{3}\rho_2(a)^3\rho_3(b) - \frac{8}{3}\rho_3(a)\rho_2(b)^3 - 2\rho_3(a)\rho_4(b) - 2\rho_4(a)\rho_3(b) \\
&- \frac{32}{3}\rho_2(a)\rho_3(a)\rho_2(b)^2 - \frac{32}{3}\rho_2(a)^2\rho_2(b)\rho_3(b) - \frac{9}{2}\rho_4(a)\rho_2(b)^2 - \frac{9}{2}\rho_2(a)^2\rho_4(b) - \frac{4}{3}\rho_2(a)\rho_3(b)^2 \\
&- \frac{4}{3}\rho_3(a)^2\rho_2(b) + \frac{43}{180}\rho_2(a)\rho_2(b)^2 + \frac{43}{180}\rho_2(a)^2\rho_2(b) - 3\rho_2(a)\rho_2(b)\rho_4(a) - 3\rho_2(a)\rho_2(b)\rho_4(b) \\
&- \frac{20}{3}\rho_2(a)\rho_3(a)\rho_3(b) - \frac{20}{3}\rho_3(a)\rho_2(b)\rho_3(b) + \frac{7}{30}\rho_2(a)\rho_3(b) + \frac{7}{30}\rho_2(b)\rho_3(a) - \frac{4}{3}\rho_2(a)\rho_5(b) \\
&- \frac{4}{3}\rho_5(a)\rho_2(b) - \frac{4}{3}\rho_2(a)^2\rho_3(a)\rho_2(b) - \frac{4}{3}\rho_2(a)\rho_2(b)^2\rho_3(b)
\end{aligned}
$$

$$
\begin{aligned}
&\rho_9(ab) - \sum_{\pi \in NC(9)} \rho_\pi(a)\rho_{\mathrm{Kr}(\pi)}(b) = \\
&- \frac{21}{4}\rho_2(a)\rho_5(a)\rho_2(b) - \frac{21}{4}\rho_2(a)\rho_2(b)\rho_5(b) - \frac{21}{2}\rho_2(a)^3\rho_4(b) - \frac{21}{2}\rho_4(a)\rho_2(b)^3 - \frac{10}{3}\rho_3(a)\rho_5(b) \\
&- \frac{10}{3}\rho_5(a)\rho_3(b) - \frac{69}{4}\rho_2(a)\rho_3(a)\rho_4(b) - \frac{69}{4}\rho_4(a)\rho_2(b)\rho_3(b) + \frac{29}{40}\rho_2(a)\rho_4(b) + \frac{29}{40}\rho_4(a)\rho_2(b) \\
&- \frac{15}{4}\rho_4(a)\rho_4(b) - 6\rho_2(a)^3\rho_2(b)^3 + \frac{21}{10}\rho_2(a)^2\rho_2(b)^2 + \rho_3(a)\rho_3(b) - \frac{251}{4}\rho_2(a)\rho_3(a)\rho_2(b)\rho_3(b) \\
&- \frac{3}{4}\rho_2(a)^4\rho_2(b)^2 - \frac{3}{4}\rho_2(a)^2\rho_2(b)^4 - \frac{23}{12}\rho_2(a)^4\rho_3(b) - \frac{23}{12}\rho_3(a)\rho_2(b)^4 - \frac{173}{6}\rho_2(a)^3\rho_2(b)\rho_3(b) \\
&- \frac{173}{6}\rho_2(a)\rho_3(a)\rho_2(b)^3 - \frac{83}{4}\rho_2(a)^2\rho_2(b)^2\rho_3(a) - \frac{83}{4}\rho_2(a)^2\rho_2(b)^2\rho_3(b) - \frac{1}{6}\rho_2(a)^3\rho_3(a)\rho_2(b) \\
&- \frac{1}{6}\rho_2(a)\rho_2(b)^3\rho_3(b) + \frac{29}{60}\rho_2(a)^3\rho_2(b) + \frac{29}{60}\rho_2(a)\rho_2(b)^3 + \frac{353}{180}\rho_2(a)^2\rho_3(b) + \frac{353}{180}\rho_3(a)\rho_2(b)^2 \\
&- \frac{57}{4}\rho_2(a)^2\rho_3(a)\rho_3(b) - \frac{57}{4}\rho_3(a)\rho_2(b)^2\rho_3(b) + \frac{167}{90}\rho_2(a)\rho_3(a)\rho_2(b) + \frac{167}{90}\rho_2(a)\rho_2(b)\rho_3(b) \\
&- \frac{55}{2}\rho_2(a)^2\rho_2(b)\rho_4(b) - \frac{55}{2}\rho_2(a)\rho_4(a)\rho_2(b)^2 - \frac{9}{4}\rho_2(a)\rho_3(a)^2\rho_2(b) - \frac{9}{4}\rho_2(a)\rho_2(b)\rho_3(b)^2 \\
&- 3\rho_2(a)^2\rho_4(a)\rho_2(b) - 3\rho_2(a)\rho_2(b)^2\rho_4(b) - \frac{71}{6}\rho_2(a)\rho_4(a)\rho_3(b) - \frac{71}{6}\rho_3(a)\rho_2(b)\rho_4(b) \\
&- \frac{17}{3}\rho_3(a)^2\rho_3(b) - \frac{17}{3}\rho_3(a)\rho_3(b)^2 - \frac{25}{12}\rho_2(a)\rho_6(b) - \frac{25}{12}\rho_6(a)\rho_2(b) - \frac{19}{2}\rho_2(a)^2\rho_5(b) \\
&- \frac{19}{2}\rho_5(a)\rho_2(b)^2 - 12\rho_2(a)^2\rho_3(b)^2 - 12\rho_3(a)^2\rho_2(b)^2 - \frac{19}{4}\rho_2(a)\rho_3(b)\rho_4(b) \\
&- \frac{19}{4}\rho_3(a)\rho_4(a)\rho_2(b) - \frac{41}{1008}\rho_2(a)\rho_2(b)
\end{aligned}
$$

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