Heights of trees and depths of non-crossing partitions

Abstract

Ordered trees and non-crossing partition are important class of objects counted by the Catalan numbers. These objects arise naturally in the context of algebraic combinatorics, geometric combinatorics, topological problems, free probability theory and mathematical biology as R. Simion discussed in her paper [8]. And it is often desirable to be able to understand the expected structure of non-crossing partitions of a large size. One of key statistics required for the understanding of the asymptotic structures of non-crossing partitions is its depth. The essay discusses heights of ordered trees and depths of non-crossing partitions.

Contents

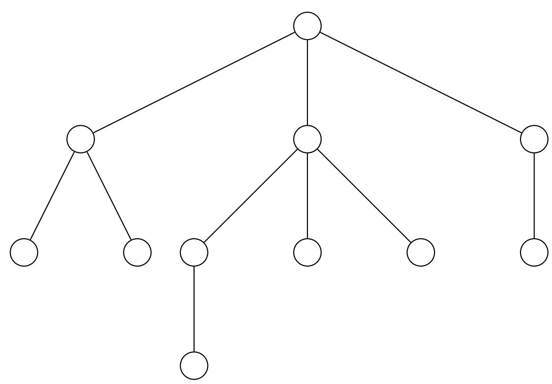
1 Introduction

The sequence of Catalan Numbers is $1, 1, 2, 5, 14, 42, \ldots$, and the formula for the n-th Catalan number is $\frac{1}{n+1}\binom{2n}{n}$. The sequence of Catalan Numbers is ubiquitous in the study of Combinatorics, and there are many combinatorial objects counted by them. Most notably, Richard Stanley listed 66 classes of objects counted by the Catalan Numbers in Volume 2 of his book [10]. And in an later issue, that number increased to 207 with the addition of "Catalan Addendum".

1.1 Planted Plane Trees

Planted plane trees are one of the class of objects counted by the Catalan numbers. They are simply connected graphs of nodes and edges, such that they are embedded in a plane with a relative order of their subtrees.

The following is an example of such a structure (if the nodes are numbered):



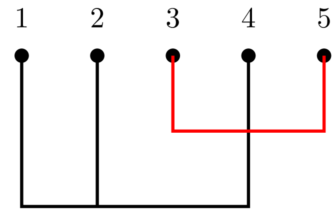
In Volume I, Chapter 2, Section 2.3 of his famous book [4], D. Knuth described the use of trees in theoretical computer science in detail. He explained how the path lengths of trees are of special importance in computation and memory allocation of computers when dealings with such structures.

In the paper [1], heights of these trees were defined as the maximal path length and their average height was considered. The main result of that paper is that the average height of a planted plane (ordered) tree with n nodes, considering all such trees to be equally likely, is

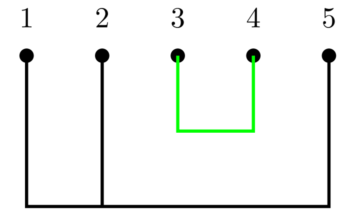
$$\sqrt{\pi n} - \frac{1}{2} + O\left(\frac{\ln n}{\sqrt{n}}\right)$$

1.2 Non-Crossing Partitions

Another important class of objects counted by Catalan numbers are the non-crossing partitions. They are essentially partitions of a set of integers with no "crossing" blocks.



A "crossing" partition with 5 nodes



A non-crossing partition with 5 nodes

These structures were first introduced in the 1972 paper by G. Kreweras [5], which prepared the stage for numerous enumerative results as well as relations between non-crossing partitions, partially ordered sets, and algebraic combinatorics.

More recently, mathematicians studying geometric group theory have become interested in non-crossing partitions. This is due to the existence of a natural, order preserving isomorphism between the non-crossing partitions and the Cayley graph of symmetric groups. J.McCammond discusses this topic in detail in his paper [12].

Another area of mathematics where non-crossing partitions have found a role in recently is the study of free probability, as well as random matrix theory. In free probability, non-crossing partitions have a critical role in describing R-transform, the free probability counter part of Fourier transform. In random matrix theory, non-crossing partitions appears naturally in the leading terms in various trace formulae. In Lecture 16 and 23 of the book [9] by A. Nica and R. Speicher, the connection between non-crossing partitions and free probability is explained in detail.

On the other hand, the recent MMath Thesis [6] of Boyu Li discusses asymptotics for some statistics on non-crossing partitions. In particular, the asymptotic distribution for the number of blocks and the number of outer blocks in non-crossing partitions are obtained.

1.3 Result of this Essay

An open problem left at the end of the thesis [6] concerns the asymptotic distribution for the depth of a random non-crossing partition.

The present essay connects non-crossing partitions to trees, and arrives to the following partial result for depths of random non-crossing partitions.

Theorem. *The average depth of a random partition in $NC(n)$ is*

$$\frac{1}{2}\sqrt{\pi n} - \frac{5}{4} - c_n + O\left(\frac{\ln n}{\sqrt{n}}\right)$$

where $0 < c_n < \frac{1}{2}$ $\forall n \in \mathbb{N}$ and $NC(n)$ is the set of all non-crossing partitions of order n.

1.4 Organization of this Essay

Many direct algorithms exist to transform a planted plane tree directly to non-crossing partitions, for example, H. Prodinger outlined two of such algorithms in his paper [7]. However, none of these algorithms present an easy way to relate the height of trees to depth of non-crossing partitions. As a result, the theorem is instead obtained by connecting $NC(n)$ to ordered trees, by using $NCP(n)$, the set of all *non-crossing pairings* of order 2n, as an intermediate class of objects. In the following sections of this essay, Section 2 presents the necessary background and notations required for this essay; Sections 3 and 4 show how one goes from $NC(n)$ to $NCP(n)$ then to $\mathcal{T}_{n+1}$, the set of *ordered trees* with $n+1$ nodes; Section 5 outlines some of the key concepts and steps in the paper [1] in obtaining the average height of ordered trees; and finally, Section 6 will showcase the proof of the previous theorem.

2 Background and Notation

2.1 Combinatorial Structures and Catalan Numbers

Definition 2.1. Fix an positive integer n,

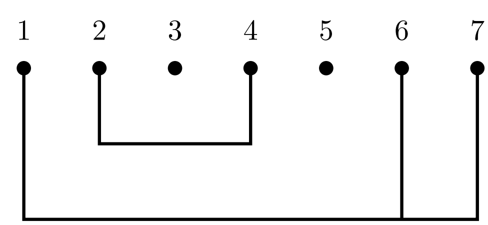
1. A *partition* π of the set $\{1, 2, \ldots, n\}$ is a collection $\{V_1, V_2, \ldots, V_k\}$, where $V_i \in \{1, 2, \ldots, n\}$ are non-empty, pairwise disjoint sets such that $V_1 \cup V_2 \cup \cdots \cup V_k = \{1, 2, \ldots, n\}$. The sets $V_1, V_2, \ldots, V_k$ are called the *blocks* of π.

2. A block of a partition with exactly 1 element is called a *singleton*; and block with more than 1 element is called a *non-singleton*.

3. A *non-crossing partition* of $\{1, 2, \ldots, n\}$ is a partition $\pi = \{V_1, V_2, \ldots, V_k\}$ of $\{1, 2, \ldots, n\}$, where for all $i \neq j$, it is not possible to have $a < b < c < d$ where $a, c \in V_i$ and $b, d \in V_j$.

4. We call a non-crossing partition of $\{1, 2, \ldots, n\}$ a *non-crossing partition of order* n. And we denote the set of all non-crossing partitions of order n by $NC(n)$.

Example 2.2.



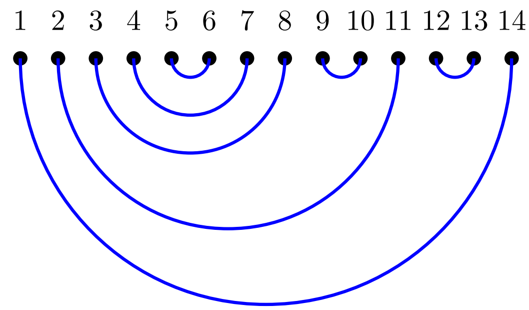
This is a picture of a *non-crossing partition* $\pi = \{\{1, 6, 7\}, \{2, 4\}, \{3\}, \{5\}\} \in NC(7)$.

Definition 2.3. Fix an positive integer n,

1. A *non-crossing pairing* π of the set $\{1, 2, \ldots, 2n\}$ is a non-crossing partition $\{(a_i, b_i)_{i=1}^n\}$ of $\{1, 2, \ldots, 2n\}$. In other words, it is a non-crossing partition of order $2n$ such that the size of each block is exactly 2.

2. We call a *non-crossing pairing* of $\{1, 2, \ldots, 2n\}$ a *non-crossing pairing of order* n. And we denote the set of all non-crossing pairings of order n by $NCP(n)$.

Example 2.4.



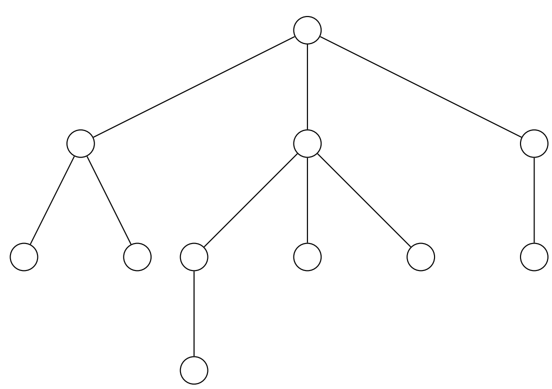
This is a *non-crossing pairing* $\rho = \{(1, 14), (2, 11), (3, 8), (4, 7), (5, 6), (9, 10), (12, 13)\} \in NCP(7)$.

Definition 2.5. Fix an positive integer n,

1. A *planted plane tree*, or an *ordered tree*, is a simply connected graph of n nodes and $n - 1$ edges, such that it is embedded in a plane so that the relative order of subtrees at each branch is a part of its structure.

2. We call a *planted plane tree* with n nodes a *tree of order* n. And we denote the set of all trees of order n by $\mathcal{T}_n$.

Example 2.6.



This is an picture of a *planted plane tree* of order 11.

Definition 2.7. The *Catalan Numbers* are a sequence of numbers $C_0, C_1, \ldots, C_n, \ldots$ such that the n-th Catalan Number, $C_n = \frac{1}{n+1}\binom{2n}{n} = \frac{2n!}{n!(n+1)!}$

Remark 2.8. It is a well-known fact that the total number of non-crossing partitions of order n, the total number of non-crossing pairings of order n, and the total number of trees of order $n+1$ are all equal to the n-th Catalan Number C_n.

Also, the generating function for Catalan Numbers is:

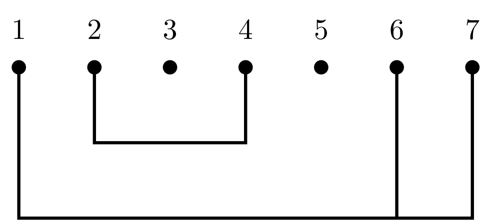
$$C(z) = \sum_{n=0}^{\infty} C_n z^n = \frac{1-\sqrt{1-4z}}{2z} \tag{2.1}$$

2.2 Depths and Heights

Definition 2.9. Let $\pi = \{V_1, V_2, \ldots, V_k\}$ be a non-crossing partition of order n.

1. For $V_i, V_j \in \pi$, we say V_i is *nested* in V_j if $max(V_i) < max(V_j)$ and $min(V_i) > min(V_j)$, and denote this relationship by $V_i \prec V_j$.
2. For $V_i, V_j \in \pi$, we say V_i is *covered* by V_j if $V_i \prec V_j$, and there does not exist $V_k \in \pi$ such that $V_i \prec V_k \prec V_j$.
3. The *depth* of $V_i \in \pi$ is defined as the following:
   (a) If V_i is an outerblock, i.e. it is not covered by another block, it has a depth of 0.
   (b) If V_i is covered by V_j, then depth of V_i = 1+ depth of V_j.
4. The *depth* of π is the maximum depth of a block in π.

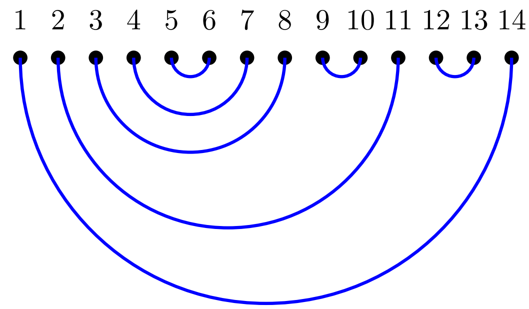
Example 2.10.



The above non-crossing partition has a depth of 2.

Definition 2.11. Similarly, the *depth* of a non-crossing pairing is the maximum depth of a block in ρ.

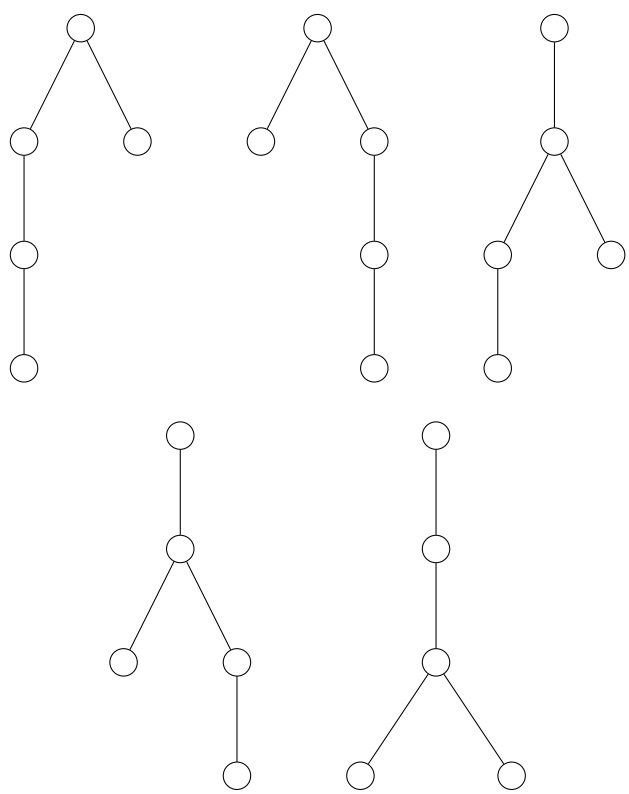
Example 2.12.



The above non-crossing pairing has a depth of 4.

Definition 2.13. The *height* of a tree is the maximal number of nodes on a simple path starting at the root.

Example 2.14.



All five planted plane trees with 5 nodes and height 4.

3 From Non-crossing Partitions to Planted Plane Trees and Back

3.1 From Non-crossing Partitions to Non-crossing Pairings and Back

Algorithm 3.1. $\forall n \in \mathbb{N}$, there exists an algorithm for converting $NC(n)$ and $NCP(n)$.

Procedure. Let $\pi = \{V_1, \ldots, V_k\} \in NC(n)$, define $\rho = \{(a_1, b_i), \ldots, (a_n, b_n)\} \in NCP(n)$ using the following procedure:

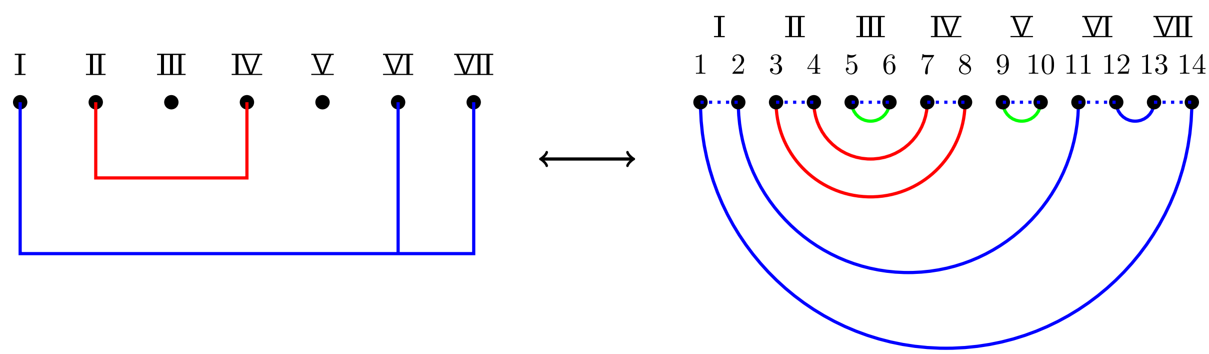
If $V_i = \{i_1, \ldots, i_{k_i}\} \in \pi$, a *non-singleton* then it corresponds to the pairs $(2i_1 - 1, 2i_k), (2i_1, 2i_2 - 1), \ldots, (2i_{k-1}, 2i_k - 1) \in \rho$.

If $V_i = \{i_1\} \in \pi$ a *singleton*, then it corresponds to a pair $(2i_1 - 1, 2i_1) \in \rho$.

Since each block of k elements in π corresponds to k pairs in ρ, and there are a total of n elements in all of the blocks in π, $\rho \in NCP(n)$.

It is an easy exercise to check that this procedure is a bijection between $NC(n)$ and $NCP(n)$. $\square$

Example 3.2.



An example of the bijection between $NC(n)$ and $NCP(n)$.

Remark 3.3. Here is an easier way to think about this bijection:

Let $\pi = \{\{\mathrm{I}, \mathrm{VI}, \mathrm{VII}\}, \{\mathrm{II}, \mathrm{IV}\}, \{\mathrm{III}\}, \{\mathrm{V}\}\}$ be a non-crossing partition of the set $\{\mathrm{I}, \ldots, \mathrm{VII}\}$. To the find the corresponding non-crossing pairing of $\{1, 2, \ldots, 14\}$, we draw line segments connecting 1 and 2, 3 and 4 ... eventually, 13 and 14. We label and think of these seven line segments as points from $\{\mathrm{I}, \mathrm{II}, \ldots, \mathrm{VII}\}$, and we try to draw arcs of pairs such that "bands" formed by the non-crossing pairing resembles the blocks of the non-crossing partition.

Proposition 3.4. *Let $\pi \in NC(n)$, $\phi_n : NC(n) \to NCP(n)$ according to the procedure in* ***3.1****, and $\mathcal{D} : \sqcup_{n=0}^{\infty} NC(n) \to \mathbb{N} \cup \{0\}$ be the function that computes the depth of non-crossing partitions, and hence also non-crossing pairings.*

1. *If $\mathcal{D}(\pi) = d$, and all the blocks in π with d nestings are singletons, then $\mathcal{D}(\phi_n(\pi)) = 2d$, where $\phi_n(\pi) \in NCP(n)$ is the corresponding non-crossing pairing of π.*

2. *If $\mathcal{D}(\pi) = d$, and at least one of the blocks in π with d nestings is a non-singleton, then $\mathcal{D}(\phi_n(\pi)) = 2d - 1$, where $\phi_n(\pi) \in NCP(n)$ is the corresponding non-crossing pairing of π.*

Proof. Because of the procedure outlined in **3.1** and **3.3**, each nesting in $\pi \in NC(n)$ produces 2 nestings in $\phi_n(\pi) \in NCP(n)$. Hence:

1. If all the blocks in π with depth d are singletons, and since a singleton in π corresponds to one pair in $\phi_n(\pi)$, then the pairs with the largest depth, or $\mathcal{D}(\phi_n(\pi)) = 2d$.

2. If at least one of the blocks in π with depth d is a non-singleton, and since a non-singleton in π corresponds to a set of pairs in $\phi_n(\pi)$ with exactly depth 1, then the pairs with the largest depth, or $\mathcal{D}(\phi_n(\pi)) = 2d + 1$.

□

Corollary 3.5. *Let $\rho \in NC(n)$, $\psi_n : NCP(n) \to NC(n)$ according to the procedure in* ***3.1****, and $\mathcal{D} : \sqcup_{n=0}^{\infty} NC(n) \to \mathbb{N} \cup \{0\}$ be as defined in* ***3.4****.*

1. *If $\mathcal{D}(\rho) = d$ is even, then $\mathcal{D}(\psi_n(\rho)) = \frac{d}{2}$.*

2. *If $\mathcal{D}(\rho) = d$ is odd, then $\mathcal{D}(\psi_n(\rho)) = \frac{d-1}{2}$.*

Proof. The results are clear from **3.4**. □

3.2 From Non-crossing Pairings to Ordered Trees and Back

Algorithm 3.6. $\forall n \in \mathbb{N}$, there exists a bijection between $NCP(n)$ and $\mathcal{T}_{n+1}$.

Procedure.

- Let $\rho = \{(a_1, b_i), \ldots, (a_n, b_n)\} \in NCP(n)$, define a tree $\tau \in \mathcal{T}_{n+1}$ using the following procedure:

  First, draw a node representing the root of τ.

  Then, for each outer pair (a pair not nested in any other pair) in ρ, put a node that is child of the root node in τ. Note that these nodes are ordered because the pairs in ρ are ordered.

  Then, for each pair nested in a outer pair, put a node that's the child of the node corresponding to that outer pair in τ.

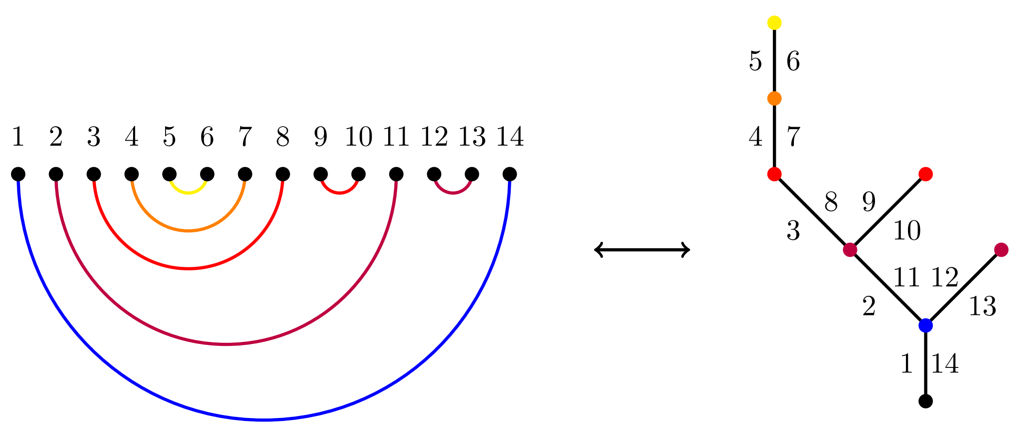
  We repeat this until all pairs in ρ are accounted for. Since each pair in ρ corresponds to one node in τ, and there are n pairs in $\rho \in NCP(n)$, we have a total of $n+1$ nodes in τ, and hence $\tau \in \mathcal{T}_{n+1}$.

- Let $\tau \in \mathcal{T}_{n+1}$, define a non-crossing pairing $\rho = \{(a_1, b_i), \ldots, (a_n, b_n)\} \in NCP(n)$ using the following procedure:

  Start to walk on the left side of the left-most edge of the root node of the tree, and label it 1. Then, keep walking around the tree on the left side, and each time a node is passed, label the next side of the edge with $i+1$, where i is the previous label. The pairs in ρ are the pairs of numbers on each side of the same edge in τ. Since there are n edges in $\tau \in \mathcal{T}_{n+1}$, $\rho \in NCP(n)$.

  It is easy to check that this procedure is a bijection between $NCP(n)$ and $\mathcal{T}_{n+1}$. $\square$

Example 3.7.



An example of the bijection between $NCP(n)$ and $\mathcal{T}_{n+1}$.

Proposition 3.8. *Let* $\rho \in NCP(n)$, $\tau \in \mathcal{T}_{n+1}$ *the tree corresponding to* ρ *by procedure in* ***3.6***, $\mathcal{D} : \sqcup_{n=0}^{\infty} NC(n) \to \mathbb{N} \cup \{0\}$ *be defined in* ***3.4***, *and* $\mathcal{H} : \sqcup_{n=1}^{\infty} \mathcal{T}_n \to \mathbb{N}$ *be a function that computes the height of a given ordered tree. Then,* $\mathcal{D}(\rho) = \mathcal{H}(\tau) - 2$.

Proof. This is clear from the procedure defined in **3.6**. □

4 Average Height of Planted Plane Trees

In this section, we will examine the paper [1] by N.G. de Bruijn, D.E. Knuth, S.O. Rice in detail and explain how they arrived at the remarkable result: the average height of planted plane trees with n nodes is

$$\sqrt{\pi n} - \frac{1}{2} + O\left(\frac{\ln n}{\sqrt{n}}\right)$$

In their paper, the definitions of planted plane trees and height are exactly the same as **2.5** and **2.13**.

Definition 4.1. Let A_{nh} be the number of trees with n nodes and height $\leq h$. Then the generating function of number of trees with height $\leq h$ is

$$A_h(z) = \sum_{h \geq 1} A_{nh} z^n \tag{4.1}$$

Remark 4.2.

1. We should note that $A_0(z) = 0$ because according to our definition of height in **2.13**, all trees has a height ≥ 1.

2. We should also note that $A_1(z) = z$ because there is only one tree of height ≤ 1, and that is the tree of only one node.

3. $A_{nh} = A_{nn} = \frac{1}{n}\binom{2n-2}{n-1} = C_{n-1} = |\mathcal{T}_n|$,the total number of trees with n nodes.

4. For any tree of height $\leq h+1$, we can think of it as taking a root node, and attaching to it any arbitrary number, possibly 0, of trees of height $\leq h$. Hence, we have

   $$A_h(z) = z \cdot [1 + A_h(z) + A_h^2(z) + A_h^3(z) + \cdots]$$

   Simplifying, we arrive at a nice recurrence relationship between $A_{h+1}(z)$ and $A_h(z)$:

   $$A_{h+1}(z) = \frac{z}{1 - A_h(z)} \tag{4.2}$$

(**4.2**) implies that $\forall h \in \mathbb{N}, A_h(z)$ are rational functions. If we define

$$A_h(z) = \frac{z \cdot p_h(z)}{p_{h+1}(z)}, \text{ where } p_h(z) \text{ are polynomials.}$$

We arrive at a linear recurrence relationship from (**4.2**):

$$\frac{z \cdot p_h(z)}{p_{h+1}(z)} = \frac{z}{1 - \frac{z \cdot p_{h-1}(z)}{p_h(z)}}$$

From **4.2.1** and **4.2.2**, we have the initial conditions for this linear recurrence: $p_0(z) = 0$ and $p_1(z) = 1$ Simplifying,

$$p_{h+1}(z) = p_h(z) - z \cdot p_{h-1}(z) \tag{4.3}$$

From (**4.3**), the characteristic polynomial for this linear recurrence is

$$r^2 = r - z \Longrightarrow r = \frac{1 \pm \sqrt{1-4z}}{2}$$

$$\Longrightarrow p_h(z) = A \cdot \left(\frac{1+\sqrt{1-4z}}{2}\right)^h + B \cdot \left(\frac{1-\sqrt{1-4z}}{2}\right)^h$$

Using the initial conditions,

$$p_0(z) = 0 = A + B;\ p_1(z) = 1 = \frac{1+\sqrt{1-4z}}{2} \cdot A + \frac{1-\sqrt{1-4z}}{2} \cdot B$$

Solving,

$$A = \frac{1}{\sqrt{1-4z}};\ B = -\frac{1}{\sqrt{1-4z}}$$

Finally,

$$p_h(z) = \frac{1}{\sqrt{1-4z}} \left[\left(\frac{1+\sqrt{1-4z}}{2}\right)^h - \left(\frac{1-\sqrt{1-4z}}{2}\right)^h\right] \tag{4.4}$$

By Cauchy's Integral Formula,

$$A_{nh} = \frac{1}{2\pi i} \int^{(0+)} A_h(z) \, \frac{\mathrm{d}z}{z^{n+1}} = \frac{1}{2\pi i} \int^{(0+)} \frac{p_h(z)}{p_{h+1}(z)} \, \frac{\mathrm{d}z}{z^n} \tag{4.5}$$

Let $u = \frac{1-\sqrt{1-4z}}{1+\sqrt{1-4z}}$. Since $u \approx z$ when $|z| \ll 1$, we can make the change of variables without changing the bounds of integration. And we have:

$$z = \frac{u}{(1+u)^2};\ 1 + u = \frac{2}{1+\sqrt{1-4z}};\ \mathrm{d}z = \frac{1-u}{(1+u)^3}\mathrm{d}u;$$

$$\frac{p_h(z)}{p_{h+1}(z)} = \frac{\left(1 - \left(\frac{1+\sqrt{1-4z}}{1-\sqrt{1-4z}}\right)\right)^h \left(\frac{1+\sqrt{1-4z}}{2}\right)^h}{\left(1 - \left(\frac{1+\sqrt{1-4z}}{1-\sqrt{1-4z}}\right)\right)^{h+1} \left(\frac{1+\sqrt{1-4z}}{2}\right)^{h+1}} = (1+u)\frac{1-u^h}{1-u^{h+1}}$$

Making the above substitutions in (**4.5**):

$$A_{nh} = \frac{1}{2\pi i} \int^{(0+)} (1-u)(1+u)^{2n-2} \frac{1-u^h}{1-u^{h+1}} \, \frac{\mathrm{d}u}{u^n} \tag{4.6}$$

If we consider $B_{nh} := A_{nn} - A_{nh}$, the number of trees with n nodes and height $> h$, some simplification of the integral in (**4.6**) will occur.

Again by Cauchy's Integral Formula,

$$A_{nn} = \frac{1}{2\pi i}\int^{(0+)} A_n(z)\,\frac{\mathrm{d}z}{z^{n+1}},\ \text{ where } A_n(z) = \sum_{n=1}^{\infty} C_{n-1}z^n = \sum_{n=0}^{\infty} C_n z^{n+1}$$

Recall (**2.1**), we have

$$A_n(z) = z\cdot\frac{1-\sqrt{1-4z}}{2z}$$

and

$$A_{nn} = \frac{1}{2\pi i}\int^{(0+)}\frac{1-\sqrt{1-4z}}{2}\,\frac{\mathrm{d}z}{z^{n+1}} = \frac{1}{2\pi i}\int^{(0+)}\frac{2}{1+\sqrt{1-4z}}\,\frac{\mathrm{d}z}{z^n}$$

Make the same change of variables and substitutions as before:

$$A_{nn} = \frac{1}{2\pi i}\int^{(0+)}(1-u)(1+u)^{2n-2}\,\frac{\mathrm{d}u}{u^n} \tag{4.7}$$

Hence, by (**4.6**) and (**4.7**),

$$\begin{aligned}
B_{nh} &:= A_{nn} - A_{nh}\\
&= \frac{1}{2\pi i}\int^{(0+)}(1-u)(1+u)^{2n-2}\left(1-\frac{1-u^h}{1-u^{h+1}}\right)\frac{\mathrm{d}u}{u^n}\\
&= \frac{1}{2\pi i}\int^{(0+)}(1-u)(1+u)^{2n-2}\left(\frac{u^h-u^{h+1}}{1-u^{h+1}}\right)\frac{\mathrm{d}u}{u^n}\\
&= \frac{1}{2\pi i}\int^{(0+)}(1-u)^2(1+u)^{2n-2}\left(\frac{u^h}{1-u^{h+1}}\right)\frac{\mathrm{d}u}{u^n}
\end{aligned}$$

Hence

$$B_{nh} = \frac{1}{2\pi i}\int^{(0+)}(1-u)^2(1+u)^{2n-2}\frac{u^{h+1}}{1-u^{h+1}}\,\frac{\mathrm{d}u}{u^{n+1}} \tag{4.8}$$

The average height of trees with n nodes is S_n/A_{nn}, where

$$S_n = \sum_{\tau\in\mathcal{T}_n} height(\tau) = \sum_{h\geq 1} h\cdot(A_{nh}-A_{n,h-1}) = \sum_{h\geq 1} h\cdot(B_{n,h-1}-B_{nh})$$

This is a telescoping sum of B_{nh}, and by (**4.8**), have:

$$S_n = \sum_{h\geq 0} B_{nh} = \frac{1}{2\pi i}\int^{(0+)}(1-u)^2(1+u)^{2n-2}\left(\sum_{h\geq 0}\frac{u^{h+1}}{1-u^{h+1}}\right)\frac{\mathrm{d}u}{u^{n+1}}$$

Since $\sum_{h\geq 0}\frac{u^{h+1}}{1-u^{h+1}}$ is the Lambert Series equaling $\sum_{k\geq 1} d(k)u^k$, where $d(k) =$ number of positive divisors of k, we have

$$S_{n+1} = \frac{1}{2\pi i}\int^{(0+)}(1-u)^2(1+u)^{2n}\left(\sum_{k\geq 1} d(k)u^k\right)\frac{\mathrm{d}u}{u^{n+2}} \tag{4.9}$$

Finally, using binomial theorem and extracting the binomial coefficients from the term $(1+u)^{2n}$ in the integral, we arrive at

$$S_{n+1} = \sum_{k\geq 1} d(k)\left[\binom{2n}{n+1-k} - 2\binom{2n}{n-k} + \binom{2n}{n-1-k}\right] \tag{4.10}$$

Eventually, we want to derive the value of $S_{n+1}/A_{n+1,n+1}$ as $n\to\infty$, where $A_{n+1,n+1} = C_n = \frac{1}{n+1}\binom{2n}{n}$.

Hence, we proceed to obtain an asymtotic series for the sum

$$f_a(n) := \sum_{k\geq 1} d(k)\cdot\left[\frac{\binom{2n}{n+a-k}}{\binom{2n}{n}}\right],\ \text{for fixed } a,\ n\to\infty \tag{4.11}$$

Note that:

$$\frac{S_{n+1}}{A_{n+1,n+1}} = (n+1)\cdot[f_1(n) - 2f_0(n) + f_{-1}(n)] \tag{4.12}$$

By Stirling's approximation, $\binom{n}{k} = \frac{n^n\sqrt{2\pi n}}{k^k\sqrt{2\pi k}(n-k)^{n-k}\sqrt{2\pi(n-k)}}$ Hence,

$$\begin{aligned}
\frac{\binom{2n}{n+a-k}}{\binom{2n}{n}} &= \frac{n^{2n}(2\pi n)}{(n+a-k)^{n+a-k}\sqrt{2\pi(n+a-k)}(n-a+k)^{n-a+k}\sqrt{2\pi(n-a+k)}}\\
&= \frac{n^{2n}(2\pi n)}{n^{n+a-k}(1-\frac{k-a}{n})^{n+a-k}\sqrt{2\pi(n+a-k)}n^{n-a+k}(1+\frac{k-a}{n})^{n-a+k}\sqrt{2\pi(n-a+k)}}\\
&= \frac{ne^{-\ln(1-\frac{k-a}{n})(n+a-k)}e^{-\ln(1+\frac{k-a}{n})(n-a+k)}}{\sqrt{(n+a-k)(n-a+k)}}\\
&= e^{-\ln(1-\frac{k-a}{n})(n+a-k)-\ln(1+\frac{k-a}{n})(n-a+k)+\ln(n)-\ln(\sqrt{(n+a-k)(n-a+k)})}
\end{aligned}$$

Now, we set $x = (k-a)/n$, and do a series expansion of the exponent terms.

When $-\frac{1}{2} < x < \frac{1}{2}$,

$$\frac{\binom{2n}{n+a-k}}{\binom{2n}{n}} = exp\left(-2n\left(\frac{x^2}{1\cdot 2} + \frac{x^4}{3\cdot 4} + \cdots\right) + \left(\frac{x^2}{2} + \frac{x^4}{4} + \cdots\right) - \frac{1}{6n}\left(x^2 + x^4 + \cdots\right) + O\left(x^2 n^{-3}\right)\right) \tag{4.13}$$

And when $k \geq n^{\frac{1}{2}+\epsilon} + a$, $\forall\epsilon > 0$, or $|x| \geq \frac{1}{2}$,

$$\frac{\binom{2n}{n+a-k}}{\binom{2n}{n}} = O(exp(-n^{2\epsilon}))$$

Now, if we define the following function

$$g_b(n) := \sum_{k\geq 1} k^b d(k) e^{-k^2/n} \tag{4.14}$$

Since the terms for $k \geq n^{\frac{1}{2}+\epsilon}$ are negligible, we can use (**4.13**) to express f_a in terms of g_b:

$$\begin{aligned}
f_a(n) =& g_0(n) + \frac{2a}{n}g_1(n) - \frac{a^2}{n}g_0(n) + \frac{4a^2+1}{2n^2}g_2(n) - \frac{1}{6n^3}g_4(n) - \frac{2a^3+a}{n^2}g_1(n)\\
&+ \frac{4a^3+5a}{3n^3}g_3(n) - \frac{a}{3n^4}g_5(n) + O(n^{-2+\epsilon}g_0(n))
\end{aligned} \tag{4.15}$$

By utilizing the Gamma and Riemann Zeta functions as well as the method of shifting residues, the paper [1] was able to derive the following estimates of $g_0(n)$, $g_1(n)$, and $g_2(n)$:

$$g_0(n) = \frac{1}{4}\sqrt{\pi n}\ln n + \left(\frac{3}{4}\gamma - \frac{1}{2}\ln 2\right)\sqrt{\pi n} + \frac{1}{4} + O(n^{-m}); \tag{4.16a}$$

$$g_1(n) = \frac{1}{4}\ln n + \frac{3}{4}\gamma n + \frac{1}{144} - \frac{1}{14400}n^{-1} + O(n^{-2}); \tag{4.16b}$$

$$g_2(n) = \frac{n}{8}\sqrt{\pi n}\ln n + \left(\frac{1}{4} + \frac{3}{8}\gamma - \frac{1}{4}\ln 2\right) n\sqrt{\pi n} + O(n^{-m}); \tag{4.16c}$$

Now, by (**4.12**), (**4.15**), and the above:

$$\begin{aligned}
\frac{1}{n+1}\left(\frac{S_{n+1}}{A_{n+1,n+1}}\right) &= f_1(n) - 2f_0(n) + f_{-1}(n) \\
&= \left(g_0(n) + \frac{2}{n}g_1(n) - \frac{1}{n}g_0(n) + \frac{5}{2n^2}g_2(n)\right) \\
&\quad - 2\left(g_0(n) + \frac{0}{n}g_1(n) - \frac{0}{n}g_0(n) + \frac{1}{2n^2}g_2(n)\right) \\
&\quad + \left(g_0(n) - \frac{2}{n}g_1(n) - \frac{1}{n}g_0(n) + \frac{5}{2n^2}g_2(n)\right) \\
&= -\frac{2}{n}g_0(n) + \frac{4}{n^2}g_2(n)
\end{aligned} \tag{4.17}$$

Substituting the results (**5.16a**), (**5.16b**), (**5.16c**),

$$\begin{aligned}
\frac{1}{n+1}\left(\frac{S_{n+1}}{A_{n+1,n+1}}\right) &= -\frac{2}{n}\left(\frac{1}{4}\sqrt{\pi n}\ln n + \left(\frac{3}{4}\gamma - \frac{1}{2}\ln 2\right)\sqrt{\pi n} + \frac{1}{4} + O(n^{-m})\right) \\
&\quad + \frac{4}{n^2}\left(\frac{n}{8}\sqrt{\pi n}\ln n + \left(\frac{1}{4} + \frac{3}{8}\gamma - \frac{1}{4}\ln 2\right) n\sqrt{\pi n} + O(n^{-m})\right) \\
&= \frac{1}{n}\sqrt{\pi n} - \frac{1}{2n} + O(n^{-\frac{3}{2}}\ln n)
\end{aligned} \tag{4.18}$$

Finally, we have the average height of planted plane trees as

$$\sqrt{\pi n} - \frac{1}{2} + O\left(\frac{\ln n}{\sqrt{n}}\right) \tag{4.19}$$

5 Average Depth of Non-crossing Partitions

In this final section, we put together the knowledge we have gained in Sections 2, 3, 4 and 5 to arrive at our main result as stated before:

Theorem 5.1. *The average depth of non-crossing partitions of order n is*

$$\frac{1}{2}\sqrt{\pi n} - \frac{5}{4} - c + O\left(\frac{\ln n}{\sqrt{n}}\right), \text{ where } 0 < c < \frac{1}{2}$$

Proof. LetD_n the random variable representing the depth of a random non-crossing partition of order n We begin with the fundamental expression for the expected depth of $\pi \in NC(n)$:

$$E(D_n) = \frac{\sum_{\pi \in NC(n)} \mathcal{D}(\pi)}{|NC(n)|} = \frac{\sum_{\pi \in NC(n)} \mathcal{D}(\pi)}{C_n} \tag{5.1}$$

By bijection in **3.1** and the results of **3.5**, (**5.1**) becomes:

$$E(D_n) = \frac{\sum_{\rho \in NCP(n), \mathcal{D}(\rho) even} \frac{\mathcal{D}(\rho)}{2}}{C_n} + \frac{\sum_{\rho \in NCP(n), \mathcal{D}(\rho) odd} \frac{\mathcal{D}(\rho)-1}{2}}{C_n}$$

$$E(D_n) = \frac{\sum_{\rho \in NCP(n)} \frac{\mathcal{D}(\rho)}{2}}{C_n} - \frac{\sum_{\rho \in NCP(n), \mathcal{D}(\rho) odd} \frac{1}{2}}{C_n} \tag{5.2}$$

Since $0 < |\{\rho \in NCP(n) | \mathcal{D}(\rho) is odd\}| < C_n$,

$$0 < \frac{\sum_{\rho \in NCP(n), \mathcal{D}(\rho) odd} \frac{1}{2}}{C_n} < \frac{1}{2}$$

. And (**5.2**) becomes:

$$E(D_n) = \frac{\sum_{\rho \in NCP(n)} \frac{\mathcal{D}(\rho)}{2}}{C_n} - c, \text{ where } 0 < c < \frac{1}{2}$$

By bijection in **3.6** and the results of **3.8**, the above becomes:

$$E(D_n) = \frac{\sum_{\tau \in \mathcal{T}_{n+1}} \frac{\mathcal{H}(\tau)-2}{2}}{C_n} - c, \text{ where } 0 < c < \frac{1}{2}$$

$$E(D_n) = \frac{1}{2} \cdot \frac{\sum_{\tau \in \mathcal{T}_{n+1}} \mathcal{H}(\tau)}{C_n} - \frac{1}{2} \cdot \frac{\sum_{\tau \in \mathcal{T}_{n+1}} 2}{C_n} - c, \text{ where } 0 < c < \frac{1}{2}$$

$$E(D_n) = \frac{1}{2} \cdot E(H_n) - 1 - c, \text{ where } 0 < c < \frac{1}{2}$$

By the final result in Section 5:

$$E(D_n) = \frac{1}{2} \cdot \left[\sqrt{\pi n} - \frac{1}{2} + O\left(\frac{\ln n}{\sqrt{n}}\right)\right] - 1 - c, \text{ where } 0 < c < \frac{1}{2}$$

$$E(D_n) = \frac{1}{2}\sqrt{\pi n} - \frac{5}{4} - c + O\left(\frac{\ln n}{\sqrt{n}}\right), \text{ where } 0 < c < \frac{1}{2} \tag{5.3}$$

□

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