SYMPLECTIC NON–KÄHLER MANIFOLDS

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Abstract. These are notes for a talk given in the Symplectic Geometry student seminar at Stony Brook on August 30, 2018. We give an example of a closed symplectic manifold not admitting a Kähler structure, published by Thurston in [1]. We then discuss the family of manifolds this example fits into, namely nilmanifolds. Determining whether a nilmanifold admits a symplectic structure is quite easy, and determining whether it admits a Kähler structure is even easier, once some results are in place. We end with an example of a nilmanifold that is complex and symplectic, all of whose odd Betti numbers are even, yet does not admit a Kähler structure.

1. Thurston's example

We present Thurston's example in [1] of a symplectic four–manifold with $b_1 = 3$. Recall that on a Kähler manifold X, we have $H^1_{dR}(X) \otimes \mathbb{C} \cong H^{1,0}_{\bar{\partial}}(X) \oplus H^{0,1}_{\bar{\partial}}(X)$ along with $H^{1,0}_{\bar{\partial}}(X) \cong \overline{H^{0,1}_{\bar{\partial}}(X)}$. From here we conclude that the first Betti number b_1 of a compact Kähler manifold is even. Similarly, all odd Betti numbers b_{2i+1} are even as well.

We start with the four–dimensional real Lie group G of matrices of the form

$$\begin{pmatrix} 1 & x & z & 0 & 0 \\ 0 & 1 & y & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & w \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

where $x, y, z, w \in \mathbb{R}$, and multiplication in the group is matrix multiplication. Note that this is a simply connected nilpotent Lie group (diffeomorphic to $\mathbb{R}^4$). We read off the left-invariant one-forms on G as the entries of

$$A^{-1}dA = \begin{pmatrix} 1 & -x & xy-z & 0 & 0 \\ 0 & 1 & -y & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -w \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & dx & z & 0 & 0 \\ 0 & 0 & dy & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & dw \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$
$$= \begin{pmatrix} 0 & dx & dz - xdy & 0 & 0 \\ 0 & 0 & dy & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & dw \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

So, a basis of left-invariant forms on G is given by $\{dx, dz - xdy, dy, dw\}$. Now, if we consider the subgroup Γ of G consisting of integer matrices, the quotient of G by left-multiplication by Γ, let us denote it G/Γ, is a closed manifold of dimension four. The one-forms we mentioned are left-invariant, and so they descend to one-forms on the quotient G/Γ. We have $d(dx) = d(dy) = d(dw) = 0$ and $d(dz - xdy) = -dxdy$, and so denoting by $\alpha, \beta, \gamma, \delta$ the descended one-forms, we have (after renaming γ to $-\gamma$)

$$d\alpha = 0, \quad d\beta = 0, \quad d\gamma = \alpha\beta, \quad d\delta = 0.$$

Now consider the two-form $\alpha\gamma + \beta\delta$. We have

$$d(\alpha\gamma + \beta\delta) = \alpha\alpha\beta = 0$$

and

$$(\alpha\gamma + \beta\delta)^2 = 2\alpha\gamma\beta\delta.$$

Since $dxdydzdw \neq 0$ at every point of G, we have that $2\alpha\gamma\beta\delta$ vanishes nowhere. Therefore $\alpha\gamma + \beta\delta$ is a symplectic form on G/Γ.

To calculate b_1, we use of result of Nomizu (which we will formally state later) that tells us that we can calculate the de Rham cohomology of G/Γ by calculating the cohomology of the graded-commutative differential graded algebra of left-invariant forms on G, namely $\Lambda(\tilde{\alpha}, \tilde{\beta}, \tilde{\gamma}, \tilde{\delta}; d)$, where $\tilde{\alpha} = dx$, $\tilde{\beta} = dy$, $\tilde{\gamma} = dz - xdy$, $\tilde{\delta} = dxdy$. We see that $H^1(\Lambda(\tilde{\alpha}, \tilde{\beta}, \tilde{\gamma}, \tilde{\delta}; d)) = \mathrm{span}(dx, dy, dw)$ and so $b_1(G/\Gamma) = 3$. Therefore G/Γ cannot admit a Kähler metric.

Remark 1.1. Topologically, this four–manifold (known now in the literature as the *Kodaira–Thurston manifold*), is obtained in the following way. Start with a two-torus $S^1 \times S^1$, and consider the principal S^1 fiber bundle over $S^1 \times S^1$ classified by $\alpha\beta \in H^2(S^1 \times S^1; \mathbb{Z})$, where α, β correspond to the volume forms on the factor circles. Denote the volume form of the fiber S^1 by γ. Over the total space E of this fiber bundle, consider the trivial S^1 principal fiber bundle (classified by $0 \in H^2(E; \mathbb{Z})$), i.e. $E \times S^1$. Denote the volume form of the fiber S^1 in this iteration by δ. This space is the Kodaira–Thurston manifold described above.

2. Nilmanifolds

The Kodaira–Thurston manifold we just considered fits into a class of easy-to-study closed manifolds called *nilmanifolds*, obtained in the following way: Take a simply connected nilpotent real Lie group G (think upper-triangular matrices). Consider the complex

$$C(G) = \oplus_{p \geq 0} C^p(G),$$

where $C^p(G)$ is the vector space of left-invariant p-forms on G.

Remark 2.1. The differential d of degree $+1$ on this complex (which we encountered before) is in fact dual to the Lie bracket $[-.-]$ on left-invariant vector fields, in the following sense. If α is a left-invariant one-form, and X, Y are left-invariant vector fields, then

$$d\alpha(X, Y) = X(\alpha(Y)) - Y(\alpha(X)) - \alpha([X, Y]) = -\alpha([X, Y]).$$

(The terms $\alpha(Y)$ and $\alpha(X)$ are constant functions since α, X, Y are left-invariant, and so the corresponding terms vanish.) Denoting by $\mathfrak{g}$ the Lie algebra to G and identifying its dual $\mathfrak{g}^*$ with $C^1(G)$, we see that $C^p(G)$ can be identified with $\Lambda^p\mathfrak{g}^*$, and so the differential on $C(G)$ is determined by what it does on $C^1(G)$ upon being extended to all of $C(G)$

by the Leibniz rule. The fact that $d^2 = 0$ follows from "dualizing" the Jacobi identity on left-invariant vector fields.

Now, suppose we have a discrete subgroup Γ of G such that G/Γ is a closed manifold (where G/Γ denotes the quotient by the action of left multiplication). Then we have the following theorem:

Theorem 2.2. *(Nomizu) The map of graded-commutative differential graded algebras* $(C(G), d) \to (\Omega_{deRham}(G/\Gamma), d)$ *obtained by projection induces an isomorphism on cohomology.*

This theorem justifies the calculation of $b_1 = 3$ for the Kodaira–Thurston manifold done in the previous section.

The question is now: when can we find such a discrete subgroup Γ? If our group G really is a matrix group, then we can take those matrices with integer entries to be Γ. However, in general the existence of such a Γ in a simply connected nilpotent real Lie group is a somewhat delicate issue. We have the following useful result:

Theorem 2.3. *(Malcev) The simply connected real Lie group G admits a discrete subgroup Γ such that G/Γ is a closed manifold is and only if a basis of the Lie algebra $\mathfrak{g}$ of G can be chosen in which the structure coefficients are rational numbers.*

Recall, the structure coefficients of a basis $\{X_1, \ldots, X_n\}$ for a Lie algebra $\mathfrak{g}$ are the coefficients in the expressions $[X_i, X_j] = \sum_k c_{ij}^k X_k$. The same coefficients show up in the expression $dx_i = -\sum_k c_{ij}^k x_j x_k$, where $x_i \in \mathfrak{g}^*$ is dual to X_i.

A nilpotent Lie group is one for which we can choose a basis $\{x_1, \ldots, x_n\}$ of $\mathfrak{g}^*$ such that dx_i is a linear combination of $dx_j dx_k$ such that $j, k < i$. Combining this with Malcev's theorem, we obtain the following simple recipe for producing a nilmanifold:

Corollary 2.4. *To assign a nilmanifold, one writes down a graded-commutative differential graded algebra generated by degree 1 variables x_i, $\Lambda(x_i)$, equipped with a differential d satisfying $dx_i = \sum_{j,k<i} c_{ij}^k x_j x_k$, with all c_{ij}^k rational. That is, for such a differential algebra, there exists a simply connected nilpotent real Lie group whose complex of left-invariant forms is given by the algebra, and the Lie group admits a discrete subgroup such that the quotient by left multiplication is a nilmanifold whose de Rham complex is quasi-isomorphic to the given algebra (tensored with the reals).*

Remark 2.5. The above usage of "assign" is a bit vague. The *rational homotopy type* of a nilmanifold is determined by the isomorphism type of $(\Lambda(x_i), d)$ as a rational differential graded algebra. A given rational homotopy type can contain multiple homotopy types of nilmanifolds within it, but we will not let that concern us right now, since all the properties we will want of our examples will depend only on their rational homotopy type.

Example 2.6. The Kodaira–Thurston nilmanifold is given by $\Lambda(x, y, z, w; d)$, where $dx = dy = dw = 0$, $dz = xy$. Another famous example of a nilmanifold is the Heisenberg nilmanifold, given by $\Lambda(x, y, z; d)$, where $dx = dy = 0$, $dz = xy$. Note that the Kodaira–Thurston manifold is the Heisenberg manifold crossed with a circle.

Remark 2.7. If we have an operator d on $\Lambda(x_1,\dots,x_n)$ taking linear expressions to quadratic expressions, and we extend it by the Leibniz rule to act on all elements of the algebra, then in order to satisfy $d^2=0$, the operator only has to satisfy d^2 on the generators x_i. For example, on quadratic expressions we have

$$\begin{aligned} d^2(x_ix_j) &= d((dx_i)x_j - x_idx_j) \\ &= (d^2x_i)x_j + (dx_i)(dx_j) - (dx_i)(dx_j) + x_id^2(x_j) = 0. \end{aligned}$$

On higher order expressions we check that $d^2=0$ by induction on wordlength.

3. Symplectic nilmanifolds

Now that we have a handy description of nilmanifolds in terms of differential graded algebras $\Lambda(x_1,\dots,x_n)$ generated in degree 1, we can ask if the algebra can tell us whether the manifold admits a symplectic form. We have the following simple criterion found in [2]:

Proposition 3.1. *Let G/Γ be a nilmanifold, and consider the quasi-isomorphism of differential graded algebras*

$$(C(G),d) \xrightarrow{\phi} (\Omega_{dR}(G/\Gamma),d).$$

The nilmanifold G/Γ admits a symplectic structure if and only if there is a closed element $\omega \in C^2(G)$ such that $\omega^n \neq 0$ (where $2n$ is the dimension of the Lie group and corresponding nilmanifold).

Proof. If $\omega \in C^2(G)$ satisfies $d\omega = 0$ and $\omega^n \neq 0$, then $\phi(\omega)$ satisfies $d\phi(\omega) = \phi(d\omega) = 0$. Denoting by $x_1,\dots,x_n$ a basis of $C^1(G)$, we have $\omega^n = ax_1x_2\dots x_n$ with $a \neq 0$, and so $(\phi(\omega))^n = \phi(\omega^n) = a\phi(x_1)\cdot\dots\cdot\phi(x_n)$ which is nowhere zero as we saw on the example of the Kodaira–Thurston manifold.

Conversely, if ω is a symplectic form on G/Γ, then there is a class $[\tilde\omega] \in H^2(C(G),d)$ such that $\phi^*([\tilde\omega]) = [\omega]$. (Note that ϕ is not necessarily invertible as a differential graded algebra map, so we cannot directly apply the same reasoning as in the previous paragraph for the converse.) Then $[\tilde\omega]^n \neq 0$ since $\phi^*([\tilde\omega]^n) = [\omega^n] \neq 0$, so in particular $\tilde\omega^n \neq 0$. □

Example 3.2. Writing the Kodaira–Thurston manifold as $\Lambda(x,y,z,w; dx = dy = dw = 0, dz = xy)$, we see that the manifold is symplectic since the element $xz+yw$ satisfies $d(xz+yw)=0$ and $(xz+yw)^2 = 2xzyw \neq 0$.

4. Kähler nilmanifolds and formality

To conclude that a (nil)manifold does not admit a Kähler metric, it suffices to show that some odd Betti number b_{2i+1} is odd. But more is true in the case of nilmanifolds; nilmanifolds are almost never Kähler:

Theorem 4.1. *If a nilmanifold G/Γ admits a Kähler metric, then G is abelian and G/Γ is (diffeomorphic to) a torus.*

The argument will pass through Sullivan–Quillen rational homotopy theory. In rational homotopy theory (let us in fact restrict to the *real homotopy theory* of manifolds for simplicity), one takes the de Rham complex of a manifold (Ω_{dR},d) and wishes to find a *nilpotent* differential graded algebra (Λ, d) with a quasi-isomorphism, i.e. a differential graded algebra map inducing an isomorphism on cohomology, from (Λ, d) to (Ω_{dR}, d). By *nilpotent* we mean that (Λ, d) is free as an algebra, and we can place an ordering on the generators in each degree such that dx is in the subalgebra generated by generators preceding x in the ordering, without linear terms. We call (Λ, d) the *minimal model* of the manifold. Any two minimal models are isomorphic as differential graded algebras, and contain a good deal of information about the manifold.

The story of nilmanifolds is catered to rational homotopy theory: by Nomizu's theorem, $(C(G), d)$ is a minimal model of the nilmanifold G/Γ.

A popular question in rational homotopy theory is whether the minimal model of a manifold can be obtained from the cohomology ring of the manifold. That is, given a manifold M with minimal model (Λ, d), we can ask whether there exists a quasi-isomorphism $(\Lambda, d) \xrightarrow{\sim} (H^*(M), 0)$, where the cohomology ring is interpreted as a differential graded algebra with trivial differential 0. If such a quasi-isomorphism exists, we say the manifold is *formal*. (Note that the algebra $H^*(M)$ is isomorphic to $H^*(\Lambda, d)$, and so we could replace $H^*(M)$ by $H^*(\Lambda, d)$ in the definition of formality.)

Example 4.2. We show that the Kodaira–Thurston manifold is *not* formal. Indeed, suppose there were a quasi-isomorphism $\Lambda(x, y, z, w; dz = xy) \xrightarrow{f} H^*(\Lambda(x, y, z, w; dz = xy)$. Then, since f induces an isomorphism on H^1, and $[x], [y], [w]$ span $H^1(\Lambda(x, y, z, w), d)$, we have

$$\begin{aligned} f(x) &= \alpha_1[x] + \alpha_2[y] + \alpha_3[w], \\ f(y) &= \beta_1[x] + \beta_2[y] + \beta_3[w], \\ f(z) &= \gamma_1[x] + \gamma_2[y] + \gamma_3[w], \end{aligned}$$

where not all of the α_i are 0, and likewise for the β_i. Consider now the equation $f(dz) = df(z)$. On the one hand,

$$\begin{aligned} f(dz) = f(xy) &= f(x)f(y) \\ &= \alpha_1\beta_3[x][w] + \alpha_2\beta_3[y][w] - \beta_1\alpha_3[x][w] - \beta_2\alpha_3[y][w] \\ &= \alpha_1\beta_3[xw] + \alpha_2\beta_3[yw] - \beta_1\alpha_3[xw] - \beta_2\alpha_3[yw]. \end{aligned}$$

We used the fact that $[x][y] = 0$ since $xy = dz$. On the other hand, $df(z) = 0$ since the differential in H^* is trivial. From here we conclude that $(\alpha_1, \alpha_2, \alpha_3)$ and $(\beta_1, \beta_2, \beta_3)$ are linearly dependent. It follows that the induced map f^* on H^1 is at most rank 2, a contradiction.

The above direct argument to show non-formality can be generalized to obtain the following result:

Theorem 4.3. *(Hasegawa) A nilmanifold G/Γ is formal if and only if the differential d in $\Lambda(x_1, \ldots, x_n; d)$ is trivial, i.e. G is abelian, i.e. G/Γ is a torus.*

Remark 4.4. We can think of formality of manifolds as the following: If a manifold is formal, then no cohomology class is represented by a form in the ideal of non-closed forms. (For the formal statement and converse, see [3]). Then we can immediately see that the Kodaira–Thurston nilmanifold $\Lambda(x, y, z, w; dz = xy)$ is not formal, since the cohomology class $[xz]$ is represented by xz, a form in the ideal of non-closed forms, i.e. the ideal generated by z. Similarly all nilmanifolds with $d \neq 0$ are not formal.

The connection between formality and admitting a Kähler metric is given by the following famous theorem [3]:

Theorem 4.5. *(Deligne–Griffiths–Morgan–Sullivan) Compact Kähler manifolds are formal.*

Example 4.6. Using all of this, we now give an example of a six–dimensional nilmanifold G/Γ which is symplectic, complex, has b_1, b_3, b_5 even, yet does not admit a Kähler metric. Consider the nilmanifold given by $\Lambda(x_1, x_2, y_1, y_2, z, w; d)$ with $dx_1 = dx_2 = dy_1 = dy_2 = 0$, $dz = x_1y_1$, and $dw = x_2y_2$. The form $\omega = x_1z + x_2w + y_1y_2$ satisfies $d\omega = 0$ and $\omega^3 \neq 0$, and so we have a symplectic form on G/Γ.

To define a complex structure on G/Γ, note that the left-invariant vector fields X_1, X_2, Y_1, Y_2, Z, W on G dual to the left-invariant one-forms corresponding to x_1, x_2, y_1, y_2, z, w descend to G/Γ to give a trivialization of the tangent bundle. (Denote the descended vector fields by the same letters.) We can thus define an almost complex structure J on these global basis vectors, and check that the Nijenhuis bracket vanishes at a point to conclude that it vanishes everywhere and hence the almost complex structure J is integrable.

Dualizing the differential d, we see that

$$[X_1, Y_1] = -Z, \quad [X_2, Y_2] = -W,$$

and all other brackets are 0. Define an almost complex structure J on G/Γ by setting $JX_1 = Y_1, JX_2 = Y_2, JY_1 = -X_1, JY_2 = -X_2, JZ = W, JW = -Z$. Furthermore, this almost complex structure is integrable, since we can immediately see that the Nijenhuis tensor

$$N(A, B) = [A, B] + J[JA, B] + J[A, JB] - [JA, JB]$$

vanishes identically.

As for the Betti numbers, $b_1 = 4$ since $H^1(\Lambda(x_1, x_2, y_1, y_2, z, w; d))$ is spanned by $[x_1], [x_2], [y_1], [y_2]$, and so by duality $b_5 = 4$. Direct calculation in the minimal model gives us that H^3 is spanned by

$$[x_1y_1z], [x_1x_2z], [x_1x_2w], [x_1y_2z], [x_1y_2w], [y_1x_2z], [y_1x_2w], [y_1y_2z], [y_1y_2w], [x_2y_2w],$$

and so $b_3 = 10$.

This nilmanifold does not admit a Kähler structure since the differential in its minimal model is non-trivial.

Example 4.7. The Kodaira–Thurston nilmanifold $\Lambda(x, y, z, w; dz = -xy)$ admits a complex structure as well. Denoting the vector fields dual to x, y, z, w by X, Y, Z, W we can set $JX = Y$ and $JZ = W$. A direct check of the Nijenhuis tensor confirms that this almost complex structure is integrable.

References

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[3] Deligne, P., Griffiths, P., Morgan, J. and Sullivan, D., 1975. Real homotopy theory of Kähler manifolds. Inventiones mathematicae, 29(3), pp.245-274.