Remark on the DGMS formality criterion

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Abstract

In this expository note, we discuss how the formality criterion for a minimal cdga given in [2, Theorem 4.1] does not hold for all minimal cdga's. A natural condition to impose on a minimal cdga for the criterion to hold is that it is *normal*; we give a proof of validity of the criterion in this case.

Recall the characterization of formality for a minimal cdga in [2, Theorem 4.1]: $M = (\Lambda V, d)$ is formal if and only if there is a choice of vector space complements N_i to the vector spaces C_i spanned by the degree i closed generators in V_i such that every closed element in the ideal $I(\oplus_i N_i)$ is exact. We comment on this criterion and provide details for convenience; no originality is claimed.

In the above form, the "only if" part of the formality criterion does not hold for all minimal $(\Lambda V, d)$ with generating vector space $V = \oplus_i V_i$ – the condition depends on the presentation of the minimal cdga, i.e. on the choice of V_i. The statement holds for minimal cdga's which are in *normal form* [1, Section 3], [3, Definition 1.2]. We say a minimal cdga $(\Lambda V, d)$ is in normal form, or *normal*, if $d(v) \in d(\Lambda^{\geq 2} V)$ implies $dv = 0$ for $v \in V$; equivalently, there exists a splitting

$$\Lambda^{\geq 2} V = d(\Lambda^{\geq 2} V) \oplus K,$$

such that $dv \in K$ for all $v \in V$.

To illustrate this point concretely, consider the minimal cdga

$$\left(\Lambda(x_2, y_3, z_5), dy = x^2, dz = x^3\right),$$

which is the motivating example given in [3] of a non-normal minimal cdga. We necessarily have $N_3 = V_3 = \text{span}(y)$ and $N_5 = V_5 = \text{span}(z)$. Then $z - xy$ is a closed and non-exact element in the ideal $I(\oplus_i N_i)$. Meanwhile, the cdga is formal, being isomorphic to the minimal model of $S^2 \times S^5$, namely $\left(\Lambda(x'_2, y'_3, z'_5), dy' = x'^2, dz' = 0\right)$ via $x \mapsto x', y \mapsto y', z \mapsto z' + x'y'$.

The existence of complements N_i as described in the formality criterion implies formality, as argued in [2]. The step where the argument for the converse generally does not go through is the following: suppose $(\Lambda V, d)$ is formal, so we have a quasi-isomorphism ψ to its cohomology. Then C_i, the subspace of closed elements in V_i, injects into the cohomology, and N_i is defined to be the kernel of ψ restricted to V_i.

Generally one does not have $V_i = C_i \oplus N_i$. In the above example, we see that $\psi(x) = [x]$ and $\psi(y) = 0$ necessarily, and so $\psi(z)$ is non-zero. Then both C_5 and N_5 are trivial, whereas V_5 is one-dimensional.

Minimal models in the form that one would usually build them, i.e. inductively by degree (adding cohomology and killing relations), are in normal form. Furthermore, every minimal cdga M is isomorphic to a normal minimal cdga [1, Section 3]: namely, take any presentation $M = (\Lambda V, d)$. Set K to be any vector space complement to $d(\Lambda^{\geq 2} V)$ in $\Lambda^{\geq 2} V$. Denote by C_i the subspace of closed elements in V_i, and choose a basis $\{n_j\}$ for a complement to C_i in V_i. For one of these complementary basis elements n_j, we have $dn_j = u + v$, where $u \in d(\Lambda^{\geq 2} V)$ and $v \in K$. Choose $u' \in \Lambda^{\geq 2} V$ such that $du' = u$ and set $n_j' = n_j - u'$. This relabelling changes the V_i, but does not change $\Lambda^{\geq 2} V$. Note that the subspace of closed elements in V_i will generally increase under this procedure.

Now, for a normal minimal cdga $A = (\Lambda V, d)$, formality implies the Deligne–Griffiths–Morgan–Sullivan splitting criterion: concretely, fix a choice of complement K, and take a quasi-isomorphism ψ from $(\Lambda V, d)$ to its cohomology. As in the argument in [2, Theorem 4.1]., note that the subspace C_i of closed elements injects into the cohomology, and denote by N_i the kernel of ψ restricted to V_i. Note that $C_i \cap N_i = \{0\}$, and choose N_i' so that $V_i = C_i \oplus N_i \oplus N_i'$. Now, we have

$$(\Lambda V)_i = V_i \oplus (\Lambda^{\geq 2} V)_i = C_i \oplus N_i \oplus N_i' \oplus (\Lambda^{\geq 2} V)_i.$$

If $c + n + p \in (\Lambda V)_i$ is closed, where $c \in C_i, n \in N_i \oplus N_i', p \in (\Lambda^{\geq 2} V)_i$, then since the cdga is normal, $dn = dp = 0$.

Due to formality, anything in the image of ψ is also mapped to by a closed element in A. Indeed, since the induced map ψ^* is an isomorphism, for any $b \in A$ we have $\psi(b) = \psi^*([a])$ for some closed $a \in A$. Now suppose we have an element $n' \in N_i'$ such that $\psi(n') \neq 0$. By the above, $\psi(n') = \psi(c + p)$ for some closed elements $c \in C_i, p \in (\Lambda^{\geq 2} V)_i$.

Relabelling n' to denote $n' - p$, we have $\psi(n') = \psi(c)$, i.e. C_i surjects onto the image of ψ. Note, we have again changed V_i (but not $\Lambda^{\geq 2} V$), so we have a commutative diagram

$$\begin{array}{ccc} (\Lambda W, d) & & \\ \cong \uparrow f & \overset{\tilde{\psi}}{\searrow} & \\ (\Lambda V, d) & \xrightarrow{\psi} & H(A) \end{array}$$

where $(\Lambda W, d)$ is also normal (with the transported complement K; we use the same notation C_i and N_i for the analogously constructed subspaces of W_i; these coincide with the images under f of C_i and N_i in V_i; we also denote $f(N_i')$ by N_i'). Then $\tilde{\psi}$ maps C_i isomorphically onto the image of $\tilde{\psi}|_{W_i}$. On the other hand, $\tilde{\psi}$ maps $W_i / \ker(\psi) = C_i \oplus N_i'$ isomorphically onto the image of $\tilde{\psi}|_{W_i}$. Therefore we have that the inclusion of C_i into $C_i \oplus N_i'$ is an isomorphism; therefore $N_i' = \{0\}$. In particular, in $(\Lambda V, d)$, we have $V_i = C_i \oplus N_i$. Then any closed element in the ideal $I(\oplus_i N_i)$ is exact as argued in [2].

To summarize, we have: *Let* $M = (\Lambda V, d)$*. If there is a choice of complement* N_i *in* V_i *to the subspace* C_i *of closed elements such that any closed element in the ideal* $I(\oplus_i N_i)$ *is exact, then* M *is formal. If* $(\Lambda V, d)$ *is a normal minimal cdga which is formal, then such a splitting* $V_i = C_i \oplus N_i$*, for all* i*, exists. If* $(\Lambda V, d)$ *is minimal but not normal, such a splitting need not exist. Every minimal cdga is isomorphic to a normal minimal cdga.*

References

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