# Topological aspects of complex and almost complex manifolds

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## Complex manifolds

A **complex manifold** is a smooth manifold equipped with a holomorphic atlas, i.e. a system of charts for which the transition functions are holomorphic.

This gives us holomorphic coordinates $(z_1, \ldots, z_n) = (x_1 + iy_1, \ldots, x_n + iy_n)$, which in turn gives us a natural operator $J$ on vector fields:

$$J\frac{\partial}{\partial x_1} = \frac{\partial}{\partial y_1}, \quad J\frac{\partial}{\partial y_1} = -\frac{\partial}{\partial x_1}, \ldots$$

(well defined globally by the Cauchy–Riemann equations). Notice, $J^2 = -\mathrm{Id}$.

## Almost complex manifolds

An **almost complex manifold** is a smooth manifold equipped with an endomorphism $J$ of its tangent bundle, satisfying $J^2 = -\mathrm{Id}$ (called an **almost complex structure**).

A natural question to ask is, given an almost complex structure, does it come from holomorphic charts like before (i.e., is it **integrable**)?

**Theorem (Newlander–Nirenberg)**

*An almost complex structure $J$ is induced by holomorphic charts if and only if its Nijenhuis tensor*

$$N_J(X, Y) = [X, Y] + J[JX, Y] + J[X, JY] - [JX, JY]$$

*vanishes, for all vector fields $X, Y$.*

**Example:** A non-integrable almost complex structure.

Consider the Lie group of real matrices of the form

$$\begin{pmatrix} 1 & x & z & 0 & 0 \\ 0 & 1 & y & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & w \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Its Lie algebra is spanned by $X, Y, Z, W$ satisfying $[X, Y] = Z$, and all other brackets among basis vectors vanish. These extend to global vector fields, which are a basis of the tangent space $T_g G$ at every point, by (left) translation.

We can define an almost complex structure by setting

$$\begin{aligned} JX &= Z, \quad JZ = -X, \\ JY &= W, \quad JW = -Y. \end{aligned}$$

Then $N_J(X, Y) = Z$, so this $J$ is not integrable.

On the other hand, $J$ defined by $JX = Y$ and $JZ = W$ is integrable.

One can obtain compact (almost) complex manifolds from this Lie group by taking the quotient by the discrete subgroup of integer matrices.

This leads one to the question:

Given an almost complex manifold, is there a complex structure on the manifold (possibly inducing a different almost complex structure)?

This question is **wide open** for compact manifolds of real dimension $\geq 6$, known sometimes as *Yau's challenge*.

The most famous case of this question is the six-dimensional sphere $S^6$.

## Almost complex structures on $S^6$

We can think of the six-sphere as the unit sphere in the imaginary octonions $\mathbb{R}^7$.

With this picture, we obtain an almost complex by declaring $J$ at a point $p$ to be (left) multiplication by the octonion $p$ on a tangent vector $v$ at $p$ (which can also be naturally thought of as an octonion):

$$J_p(v) = pv.$$

In fact, using the isometry group $SO(7)$ of $S^6$ with the Riemannian metric it inherits from Euclidean space, we can obtain other "octonionic" almost complex structures. Indeed, for $A \in SO(7)$ consider

$$(AJ)_p(v) = A^{-1} J_{Ap}(Av).$$

The stabilizer of this action is the Lie group of real algebra automorphisms of the octonions, known as $G_2$, and hence the orbit of the above $J$ under this $SO(7)$ action is $SO(7)/G_2 = \mathbb{RP}^7$.

All of these almost complex structures have non-vanishing Nijenhuis tensor, which can be seen via direct computation or via more indirect arguments.

The inclusion $\mathbb{RP}^7 \hookrightarrow \text{all almost complex structures on } S^6$ captures the fundamental group and all other homotopy groups modulo torsion (Ferlengez–Granja–M.), which we will discuss later.

What is the situation in low dimension?

First of all, any almost complex manifold has a canonical orientation (once it is fixed for all time how $\mathbb{C}$ inherits an orientation from its complex structure). In particular, non-orientable manifolds cannot admit (almost) complex structures.

In **dimension 2**, every compact (without boundary) orientable manifold can be given the structure of a Riemann surface, i.e. that of a complex projective curve.

In **dimension 4**, there is more going on. First, there are obstructions to admitting an almost complex structure, even for orientable 4–manifolds. Then, using fundamental work of Kodaira, one can find *further* obstructions to admitting a complex structure.

## Kähler manifolds

A complex manifold with a Riemannian metric $g$ such that $g(JX, JY) = g(X, Y)$ for all $X, Y$ is called **Kähler** if the non-degenerate two–form $\omega(X, Y) = g(JX, Y)$ is closed, i.e. $d\omega = 0$.

The two–form $\omega$ being non-degenerate means that for every non-zero $X$ there is a $Y$ such that $\omega(X, Y) \neq 0$. Indeed, $\omega(X, JX) = g(JX, JX) = ||JX||^2 = ||X||^2 > 0$.

Kähler manifolds have the structure of a complex manifold, a Riemannian manifold, and a **symplectic** manifold (i.e. the existence of a closed non-degenerate two–form), which are all mutually compatible.

**Examples:** Complex projective space $\mathbb{CP}^n$ is Kähler, and any complex submanifold of a Kähler manifold is also Kähler.

There are **strong topological obstructions** to admitting a Kähler structure for compact manifolds.

## Topological obstructions to Kähler structures

Let us take a step back, and consider again general compact complex manifolds.

The complex-valued smooth differential forms on a complex manifold are of the form

$$f\, dz_{i_1} \wedge \cdots dz_{i_k} \wedge d\bar{z}_{j_1} \wedge \cdots \wedge d\bar{z}_{j_l},$$

where $f$ is a smooth function in $z_i, \bar{z}_i$. Then the de Rham differential $d$ decomposes as

$$d = \partial + \bar{\partial},$$

where

$$\partial(f\, dz_1 \wedge \cdots dz_k \wedge d\bar{z}_1 \wedge \cdots \wedge d\bar{z}_l) = \sum_i \frac{\partial f}{\partial z_i} dz_i \wedge dz_{i_1} \wedge \cdots dz_{i_k} \wedge d\bar{z}_{j_1} \wedge \cdots \wedge d\bar{z}_{j_l}$$

and

$$\bar{\partial}(f\, dz_1 \wedge \cdots dz_k \wedge d\bar{z}_1 \wedge \cdots \wedge d\bar{z}_l) = \sum_i \frac{\partial f}{\partial \bar{z}_i} d\bar{z}_i \wedge dz_{i_1} \wedge \cdots dz_{i_k} \wedge d\bar{z}_{j_1} \wedge \cdots \wedge d\bar{z}_{j_l}.$$

Since $d = \partial + \bar{\partial}$, we have $0 = d^2 = \partial^2 + \partial\bar{\partial} + \bar{\partial}\partial + \bar{\partial}^2$. We can thus consider the **Dolbeault cohomology**

$$H_{\bar{\partial}} = \frac{\ker \bar{\partial}}{\mathrm{im}\, \bar{\partial}}.$$

By harmonic theory (Hodge), the groups $H_{\bar{\partial}}^{p,q}$ on a compact complex manifold are finite dimensional, since they are isomorphic to the kernel of the Laplacian associated to $\bar{\partial}$,

$$\Delta_{\bar{\partial}} = \bar{\partial}\bar{\partial}^* + \bar{\partial}^*\bar{\partial}$$

with respect to any metric, which is an elliptic operator.

**Symmetries in Dolbeault cohomology.**

- By *Serre duality*, for any compact complex $n$–manifold, we have an isomorphism

  $$H_{\bar{\partial}}^{p,q} \cong H_{\bar{\partial}}^{n-p,n-q}.$$

- For **Kähler** manifolds, we furthermore have $H_{\bar{\partial}}^{p,q} \cong H_{\bar{\partial}}^{q,p}$. What is more, the de Rham cohomology $\frac{\ker d}{\mathrm{im}\, d}$ is recovered from the Dolbeault cohomology by

  $$H^k_{\mathrm{de\ Rham}} \cong \bigoplus_{p+q=k} H_{\bar{\partial}}^{p,q}.$$

  In particular, we have that $H^1_{\mathrm{de\ Rham}} \cong H_{\bar{\partial}}^{1,0} \oplus H_{\bar{\partial}}^{0,1} \cong H_{\bar{\partial}}^{1,0} \oplus H_{\bar{\partial}}^{1,0}$ is **even–dimensional**.

On a compact 6–manifold (i.e. complex dimension 3):

| | | | | |
|---|---|---|---|---|
| 3 | | | $H^{2,3}_{\overline{\partial}}$ | |
| 2 | | | | |
| 1 | $H^{0,1}_{\overline{\partial}}$ | | | |
| 0 | | $H^{1,0}_{\overline{\partial}}$ | | |
| | 0 | 1 | 2 | 3 |

Now, there is the monumental theorem of Kodaira:

**Theorem (Kodaira)**

*A compact complex surface with even-dimensional* $H^1_{\mathrm{deRham}}$ *admits a Kähler structure.*

Using this, we can create almost complex 4–manifolds that do not admit any complex structure.

A compact almost complex 4–manifold with no complex structure.

Consider the manifold

$$M = (S^2 \times S^2)\#(S^1 \times S^3)\#(S^1 \times S^3).$$

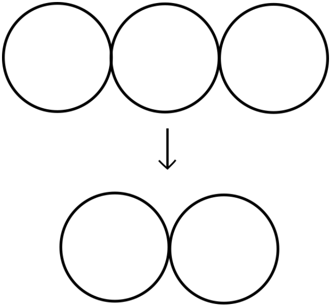
It admits an almost complex structure by classical obstruction theory (Wu).

Now, suppose it had a complex structure. Then, since $H^1_{\mathrm{deRham}}$ is 2–dimensional, it would have a Kähler structure.

Next, notice that the fundamental group $\pi_1(M)$ of this manifold is the free product $\mathbb{Z} * \mathbb{Z}$.

This group contains an index two subgroup isomorphic to $\mathbb{Z} * \mathbb{Z} * \mathbb{Z}$.

Indeed, consider the double cover:



given by rotation by $180^\circ$. This tells us there is a double cover $\widetilde{M}$ of $M$ with fundamental group

$$\pi_1(\widetilde{M}) \cong \mathbb{Z} * \mathbb{Z} * \mathbb{Z}.$$

This furthermore implies that $H^1_{\mathrm{deRham}}(\widetilde{M})$ is **3–dimensional**.

On the other hand, being a double cover of $M$, the space $\widetilde{M}$ is also a compact manifold, which inherits a Kähler structure from $M$ via pulling back by the projection $\widetilde{M} \to M$.

Therefore, $M$ could not have admitted a complex structure to begin with.

There are other methods for showing a 4–manifold does not admit a complex structure, almost all of which in some way come down to employing Kodaira's results.

Another such method makes use of another great theorem (Deligne–Griffiths–Morgan–Sullivan) that compact Kähler manifolds are *formal* in the sense of rational homotopy theory.

Using this, one can for example show that a compact quotient of the matrix group corresponding to the Lie algebra spanned by $X, Y, Z, W$ with $[X, Y] = Z, [X, Z] = W$ and all other brackets zero, does not admit a complex structure.

To reiterate Yau's challenge, there are **no** such (known) techniques in dimensions $\geq 6$.

Rephrasing: *there are no known topological obstructions to admitting a complex structure, beyond those to admitting an almost complex structure.*

## Contrast between complex and symplectic

A manifold admits an almost complex structure if and only if it admits an **almost symplectic structure**, i.e. a non-degenerate two–form $\omega$ (not necessarily satisfying $d\omega = 0$). Indeed, choose a metric $g$ compatible with $J$, and define $\omega(X, Y) = g(JX, Y)$.

The integrability, i.e. "local triviality", condition in the symplectic setting is that $d\omega = 0$. Then there are charts for the manifold in which the symplectic form has a standard form.

In contrast to almost complex and complex, the passage from almost symplectic to symplectic **is known to yield new topological constraints**.

Since the symplectic form is non-degenerate, on a $2n$–dimensional symplectic manifold $M$, we have that $\omega^n$ is a nowhere-zero top-degree form. Therefore

$$\int_M \omega^n \neq 0,$$

which implies that the de Rham cohomology class $[\omega]^k$ is non-zero for all $0 \leq k \leq n$.

Denoting $b_i(M) = \dim H^i_{\mathrm{deRham}}(M)$ (the $i^{th}$ *Betti number*), we thus have

$$b_{2i}(M) \geq 1$$

for a symplectic $2n$–manifold $M$, where $0 \leq i \leq n$.

## Example: four–dimensional torus

Consider the four–dimensional torus $T^4$, with global nowhere-zero "angular" one–forms $dx, dy, dz, dw$.

Take $\omega = dx \wedge dy + dz \wedge dw$. Then $d\omega = 0$ (using Leibniz rule and $ddx = 0$, etc.). Also, $\omega$ is non-degenerate; this is in fact equivalent to $\omega^2 = \omega \wedge \omega$ being non-zero at every point. We have

$$\omega \wedge \omega = 2dx \wedge dy \wedge dz \wedge dw.$$

In particular,

$$\int_{T^4} \omega^2 = 2.$$

Taking the almost complex structure that sends $\frac{\partial}{\partial x}$ to $\frac{\partial}{\partial y}$ and $\frac{\partial}{\partial z}$ to $\frac{\partial}{\partial w}$, and the Riemannian metric $g = dx^2 + dy^2 + dz^2 + dw^2$ yields a Kähler structure.

$$\begin{tikzcd}
 & \text{Kähler} \arrow[dl] \arrow[dr] & \\
\text{complex} \arrow[dr, dashed] & & \text{symplectic} \arrow[dl] \\
 & \text{almost complex} = \text{almost symplectic} &
\end{tikzcd}$$

There are

- examples of almost complex manifolds with no symplectic structure (e.g. $S^6$),
- examples of complex and symplectic manifolds with no Kähler structure (e.g. the compact Lie group quotient discussed earlier, and products of that with $(S^2)^n$),
- examples of complex manifolds with no symplectic structure (e.g. the Hopf manifolds $S^1 \times S^{2n-1}$).

In dimensions $\geq 6$ there are no known examples of symplectic manifolds with no complex structure.

## Obstructions to almost complex structures

At the lowest level, we can thus ask what the topological restrictions are on compact manifolds that admit almost complex structures.

These involve characteristic classes, obstruction theory, and index theory.

One can see, for example, that the only spheres that admit almost complex structures are $S^2$ and $S^6$ (Borel–Serre).

## Characteristic classes: Stiefel–Whitney classes

Generally, **characteristic classes** are cohomology classes of a space $X$ corresponding to a vector bundle $E \to X$. Vaguely, they measure how far the bundle is from being trivial; more precisely, they are obstructions to finding tuples of linearly independent sections of the bundle.

The first characteristic classes to appear historically are the **Stiefel–Whitney classes**. Given a **real** vector bundle $E \to X$, these are $\mathbb{Z}/2$–valued cohomology classes $w_i(E) \in H^i(X; \mathbb{Z}/2)$.

The first Stiefel–Whitney class $w_1(E)$ has a simple geometric interpretation:

- The 1–skeleton of $X$, i.e. the "1–dimensional part" of $X$, for a path-connected space $X$, can be identified with a bouquet of circles.

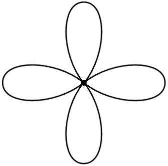
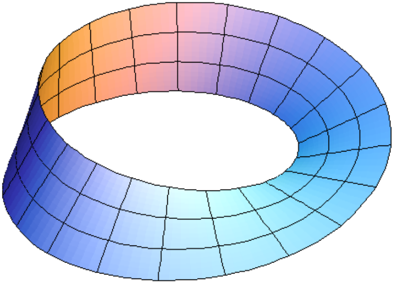


Figure: A bouquet of four circles. This is for example the 1–skeleton of the usual cell decomposition of the four–torus. (picture credit: Wikipedia)

- Take the point $x$ at the center of the bouquet, and take an ordered basis of the vector space $E_x$ at that point.
- For a circle in the bouquet, go around the circle with this ordered basis, and compare the new ordered basis at $x$ with the original.
- If the new basis has the same determinant as the original one, assign to this circle the value $+1$; otherwise assign $-1$.
- This gives us a function $\pi_1(X) \to \mathbb{Z}/2$, which corresponds to an element in $H^1(X; \mathbb{Z}/2)$; this element is $w_1(E)$. (note: we could have chosen a different cell decomposition of $X$, with a different 1–skeleton; it is a good exercise to see how the induced map $\pi_1(X) \to \mathbb{Z}/2$ does not depend on this choice)

Rephrasing, we have:

$w_1(E) = 0$ **if and only if** $E$ **is trivial when restricted to the 1–skeleton of** $X$.



**Example:** The Möbius band, thought of as a real line bundle over a circle, has $w_1 \neq 0$. (picture credit: unknown, found online)

On a manifold $M$, we can take the tangent bundle $E = TM$. Then $w_1(TM) = 0$ corresponds to the manifold $M$ being orientable.

Similarly to the $w_1$ discussion above, $w_2(E) = 0$ for an orientable bundle if and only if $E$ is trivial over the 2–skeleton of $X$, for $E$ a vector bundle of rank $\geq 3$.

## Characteristic classes: Chern classes

Associated to **complex** vector bundles, we have the **Chern classes** $c_i \in H^{2i}(X; \mathbb{Z})$.

An almost complex structure on a manifold gives its tangent bundle the structure of a complex vector bundle, so we can talk about the *Chern classes of an almost complex manifold* $c_i(M) = c_i(TM)$.

The Chern classes and the Stiefel–Whitney classes are related in the following way:

$$c_i \bmod 2 = w_{2i}.$$

Furthermore, for a complex vector bundle, $w_{2i+1} = 0$ for all $i$ (generalizing $w_1 = 0$ since a complex structure implies orientability).

The Stiefel–Whitney classes of a manifold (i.e. those of its tangent bundle) are **homotopy invariant**. That is, if $M$ and $N$ are two homotopy equivalent compact manifolds, then they have the same Stiefel–Whitney classes.

This gives us a first topological obstruction to admitting an almost complex structure (beyond orientability and even–dimensionality):

The second Stiefel–Whitney class of a manifold that admits an almost complex structure must be the mod 2 reduction of an integral class.

It is actually quite non-trivial to find compact manifolds *not* satisfying the above property. In dimensions $\leq 4$, all manifolds satisfy the above property (Wu, Thom, Whitney).

The simplest example of a manifold $M$ whose $w_2(TM)$ is not the mod 2 reduction of a class in $H^2(M; \mathbb{Z})$ is the *Wu manifold* $SU(3)/SO(3)$.

There are also manifolds all of whose $w_{2i}$ are the mod 2 reduction of an integral class, together with $w_{2i+1} = 0$ for all $i$, yet they do not admit almost complex structures; for example, the quaternionic projective plane $\mathbb{HP}^2$.

To see that $\mathbb{HP}^2$ does not admit an almost complex structure, we will need to identify some more obstructions.

## Characteristic classes: Pontryagin classes

We consider one more family of characteristic classes, associated to **real vector bundles**; these are the **Pontryagin classes** $p_i \in H^{4i}(X;\mathbb{Z})$.

We have the following relations between these and the Stiefel–Whitney and Chern classes:

- $p_i \bmod 2 = w_{2i}^2$
- If $E$ is a complex vector bundle, then

$$(1 - p_1 + p_2 - \cdots) = (1 - c_1 + c_2 - c_3 + \cdots)(1 + c_1 + c_2 + c_3 + \cdots).$$

To obtain more obstructions, we will combine this second point with a first index–theoretic invariant:

### Signature of a $4n$–dimensional compact manifold

On a $4n$–dimensional compact manifold $M$, one has the bilinear pairing

$$H^{2n}_{\mathrm{dR}}(M) \otimes H^{2n}_{\mathrm{dR}}(M) \to \mathbb{R}$$

given by

$$\alpha \otimes \beta \mapsto \int_M \alpha\beta.$$

This pairing is non-degenerate by Poincaré duality, and it is symmetric.

Choose any basis for $H^{2n}_{\mathrm{dR}}$, and consider the matrix corresponding to the above pairing. It is diagonalizable to a diagonal matrix of $+1$ and $-1$.

The difference $(\# \text{ of } +1)$ - $(\# \text{ of } -1)$ is called the **signature** $\sigma(M)$ of the manifold.

## Signature: examples

**Example 1.** Consider the manifold $S^2 \times S^2$. Choosing the Poincaré duals of $S^2 \times *$ and $* \times S^2$ as a basis for $H^2_{\mathrm{dR}}(S^2 \times S^2)$, the matrix corresponding to the above bilinear pairing is

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

We have

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$$

So the signature $\sigma(M) = 0$ vanishes.

**Example 2.** Consider the complex projective plane $\mathbb{CP}^2$. Then $H^2_{\mathrm{dR}}$ is 1–dimensional, spanned by the Poincaré dual of a complex projective line $\mathbb{CP}^1$ in $\mathbb{CP}^2$. Since two distinct lines (representing this same class) intersect in one point, the corresponding pairing matrix for $\mathbb{CP}^2$ is $(1)$. In particular, $\sigma(\mathbb{CP}^2) = 1$.

Similarly, $\sigma(\mathbb{HP}^2) = 1$.

Relating signature and Pontryagin classes

A famous theorem of Hirzebruch–Thom tells us that the signature of a compact $4n$–dimensional manifold can be obtained by integrating a certain universal rational polynomial in its Pontryagin classes.

**Theorem (Hirzebruch–Thom)**

*There is a rational polynomial $L$ in the symbols $p_1, p_2, \dots$ such that $\int_M L = \sigma(M)$.*

*The polynomial $L$ is universal, i.e. it does not depend on the manifold $M$, and it is explicitly computable:*

$$L = 1 + \tfrac{1}{3}p_1 + \tfrac{1}{45}(7p_2 - p_1^2) + \tfrac{1}{945}(62p_3 - 13p_1p_2 + 2p_1^3) + \cdots$$

## Recap of characteristic classes and properties

To summarize, we have:

- the **Stiefel–Whitney** classes $w_i(E) \in H^i(X; \mathbb{Z}/2)$ for **real** vector bundles. A bundle is orientable iff $w_1(E) = 0$. A manifold $X$ is orientable iff $w_1(TX) = 0$.
- the **Pontryagin** classes $p_i(E) \in H^{4i}(X; \mathbb{Z})$ for **real** vector bundles. Plugging the Pontryagin classes into the $L$–polynomial from before and integrating over a manifold gives the signature of the manifold.
- the **Chern** classes $c_i(E) \in H^{2i}(X; \mathbb{Z})$ for **complex** vector bundles. They satisfy
  - $c_i(E) \bmod 2 = w_{2i}$
  - $(1 - p_1 + p_2 - \cdots) = (1 - c_1 + c_2 - c_3 + \cdots)(1 + c_1 + c_2 + c_3 + \cdots)$.
  - on a closed $2n$–manifold, $\int_X c_n(TX) = \chi(X)$, where

    $$\chi(X) = \dim H^{\mathrm{even}}_{\mathrm{dR}}(X) - H^{\mathrm{odd}}_{\mathrm{dR}}(X)$$

    is the **Euler characteristic** of $X$.

## Example: the four–sphere does not admit an almost complex structure

Let us show now that $S^4$ does not admit an almost complex structure. Suppose it did; then

$$1 - p_1(TS^4) = (1 - c_1(TS^4) + c_2(TS^4)(1 + c_1(TS^4) + c_2(TS^4))$$

gives us $p_1(TS^4) = -2c_2(TS^4)$.

Since

$$\int_{S^4} c_2(TS^4) = \chi(S^4) = 2,$$

we have

$$\int_{S^4} p_1(TS^4) = -4.$$

However, by the Hirzebruch–Thom signature theorem,

$$\sigma(S^4) = \frac{1}{3}\int_{S^4} p_1(TS^4) = -\frac{4}{3},$$

which cannot be.

## Wu's theorem for almost complex four–manifolds

Earlier we invoked the following theorem of Wu in order to see that $(S^2 \times S^2)\#(S^1 \times S^3)\#(S^1 \times S^3)$ admits an almost complex structure:

**Theorem (Wu)**

*A compact 4–manifold $M$ admits an almost complex structure if and only if there is a class $c \in H^2(M;\mathbb{Z})$ such that $c$ mod 2 is $w_2(M)$, and $\int_M c^2 = 2\chi(M) + 3\sigma(M)$.*

That these conditions are necessary comes from $c_1$ reducing to $w_2$ mod 2, and

$$3\sigma(M) = \int_M p_1 = \int_M c_1^2 - 2c_2 = \int_M c_1^2 - 2\chi(M).$$

Now, $(S^2 \times S^2)\#(S^1 \times S^3)\#(S^1 \times S^3)$ has $w_2 = 0$, $\chi = 0$, and $\sigma = 0$, so we can take $c = 0$.

## Even more obstructions: the Atiyah–Singer index theorem

The **Atiyah–Singer index theorem** relates the index of certain differential operators on vector bundles over a manifold to quantities that can be obtained by integrating rational polynomials in characteristic classes over the manifold.

Particular cases are the Euler characteristic (the index of the operator $d + d^*$ from the even cotangent bundle to the odd cotangent bundle), the signature, and more:

**Theorem (Atiyah–Singer)**

*Let $M$ be an almost complex manifold, and $E \to M$ a complex vector bundle. Then*

$$\int_M ch(E)\, td(TM) \in \mathbb{Z}.$$

Here $\mathrm{ch}(E)$ is the **Chern character**

$$\mathrm{ch}(E) = \mathrm{rank}(E) + c_1 + \tfrac{1}{2}(c_1^2 - 2c_2) + \tfrac{1}{6}(c_1^3 - 3c_1c_2 + 3c_3) + \cdots$$

and $\mathrm{td}(TM)$ is the **Todd genus**

$$\mathrm{td}(TM) = 1 + \tfrac{1}{2}c_1 + \tfrac{1}{12}(c_1^2 + c_2) + \tfrac{1}{24}c_1c_2 + \cdots$$

The $L$–genus from the Hirzebruch–Thom signature theorem and the Chern character, Todd genus from the Atiyah–Singer index theorem are universal rational polynomials in characteristic classes characterized by:

- The $L$–genus $1 - \frac{1}{3}p_1 + \cdots$ is the unique ring homomorphism from oriented cobordism to $\mathbb{Z}$ such that $\int_{\mathbb{CP}^{2n}} L = 1$.
- The Chern character is the unique ring homomorphism from $K$–theory to rational cohomology sending a complex line bundle $L$ to $e^{c_1(L)}$.
- The Todd genus is the unique ring homomorphism from complex cobordism to $\mathbb{Z}$ such that $\int_{\mathbb{CP}^n} \mathrm{td} = 1$.

Obtaining explicit expressions for these comes down to complex analysis along with computations in symmetric polynomials.

### Example: $\mathbb{HP}^2$ does not admit an almost complex structure

Using the above, we can show that the quaternionic projective plane $\mathbb{HP}^2$ does not admit an almost complex structure.

For this, we will use that its cohomology ring is

$$H^*(\mathbb{HP}^2; \mathbb{Z}) \cong \mathbb{Z}[x]/(x^3),$$

where $x \in H^4(\mathbb{HP}^2; \mathbb{Z})$.

Assume that $\mathbb{HP}^2$ were almost complex. Then we would have

$$\mathrm{ch}(T\mathbb{HP}^2) = 4 - c_2 + \tfrac{1}{12}c_2^2 - \tfrac{1}{6}c_4$$

and

$$\mathrm{td}(T\mathbb{HP}^2) = 1 + \tfrac{1}{12}c_2 + \tfrac{1}{240}c_2^2 + \tfrac{1}{720}c_4.$$

Generally, there are terms involving $c_1, c_3$, but those vanish since $H^2(\mathbb{HP}^2; \mathbb{Z}) = H^6(\mathbb{HP}^2; \mathbb{Z}) = 0$.

Applying the Atiyah–Singer index theorem to $E = T\mathbb{HP}^2$, we have

$$\int_{\mathbb{HP}^2} \mathrm{ch}(T\mathbb{HP}^2)\,\mathrm{td}(T\mathbb{HP}^2) \in \mathbb{Z}.$$

Writing this out, this becomes

$$\int_{\mathbb{HP}^2} \tfrac{1}{60}\,c_2^2 - \tfrac{31}{180}\,c_4 \in \mathbb{Z}.$$

Simplifying, and using $\int_{\mathbb{HP}^2} c_4 = \chi(\mathbb{HP}^2) = 3$, we have

$$\int_{\mathbb{HP}^2} c_2^2 - 31 \in 60\mathbb{Z}.$$

Because of the structure of the cohomology ring, $c_2 = kx$, where $\int_{\mathbb{HP}^2} x^2 = \pm 1$. The choice of sign corresponds to the **orientation** that would be induced by the almost complex structure. If we really want to show that $\mathbb{HP}^2$ admits no almost complex structure inducing either orientation, we have to allow for both possibilities.

So, we have

$$\pm k^2 = 60\ell + 31.$$

If $\mathbb{HP}^2$ were to admit an almost complex structure, then 29 or 31 would have to be a quadratic residue modulo 60. However, the quadratic residues modulo 60 are $\{0, 1, 4, 9, 16, 21, 24, 25, 36, 40, 45, 49\}$.

Hirzebruch's obstruction

There is a very nice obstruction to an almost complex structure (which is now known to be a special case of the Atiyah–Singer index theorem):

**Theorem (Hirzebruch)**

*For a compact almost complex manifold of real dimension $4n$, we have*

$$\chi \equiv (-1)^n \sigma \bmod 4.$$

**Example.** Spheres $S^{4n}$ in dimensions divisible by 4 do not admit almost complex structures. Indeed, $\sigma(S^{4n}) = 0$ since $H^{2n}_{\mathrm{dR}}(S^{4n}) = \{0\}$, and $\chi(S^{4n}) = 2$.

## A remark on orientations

As we noticed in the $\mathbb{HP}^2$ example, for a given manifold, we can ask whether it admits an almost complex structure, but also whether it admits an almost complex structure *inducing a given orientation*. The answers to these two questions are generally different.

For example, take the complex manifold $\mathbb{CP}^2$. The orientation it inherits from its complex structure is such that $\int_{\mathbb{CP}^2} x^2 = +1$, where $x$ is a generator of $H^2(\mathbb{CP}^2;\mathbb{Z})$.

We can also consider $\overline{\mathbb{CP}^2}$, which denotes $\mathbb{CP}^2$ but with the *opposite orientation*. Switching the orientation switches the sign of the signature, so we have

$$\sigma(\overline{\mathbb{CP}^2}) = -1.$$

However, the Euler characteristic stays the same. Then, for $\overline{\mathbb{CP}^2}$, we have

$$\chi \not\equiv -\sigma \text{ mod } 4,$$

so it does not admit an almost complex structure (inducing the given orientation).

### Two points to keep in mind

- The polynomials in characteristic classes that are showing up are **rational**, and in fact, most of our arguments go through if we replace our manifolds with anything which only has the same rational cohomology.

  For example, the argument that $S^{4n}$ do not admit almost complex structures goes through just as well for manifolds $M$ such that $H^*(M;\mathbb{Q}) \cong H^*(S^{4n};\mathbb{Q})$.

- The information on characteristic classes that we get from the Atiyah–Singer theorem, for various vector bundles, is **all the information** one could possibly get, in a sense we will make precise later.

What we have been doing so far could be thought of under the general heading of the question:

What are the homotopy types of compact almost complex manifolds?

Considering the prevalence of rational polynomials, one is naturally led to the following relaxation of the above question:

What are the **rational** homotopy types of compact almost complex manifolds?

## Some rational homotopy theory

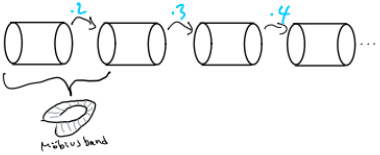
For now we mostly restrict to simply connected spaces.

For every space $X$ there is a corresponding rational space $X_{\mathbb{Q}}$ and a rationalization map $X \xrightarrow{f} X_{\mathbb{Q}}$ which induces an isomorphism on $\pi_*(-) \otimes \mathbb{Q}$ (equivalently, on $H_*(-;\mathbb{Q})$).

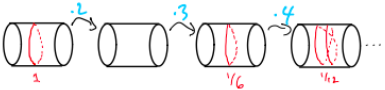
$X_{\mathbb{Q}}$ being rational means that its homotopy groups $\pi_*(X_{\mathbb{Q}})$, without tensoring with $\mathbb{Q}$, are already rational vector spaces, i.e. as abelian groups isomorphic to some $\mathbb{Q}^k$ (equivalently, $\tilde{H}_*(X_{\mathbb{Q}};\mathbb{Z})$ are rational vector spaces).

Any two rationalizations of a space are homotopy equivalent.

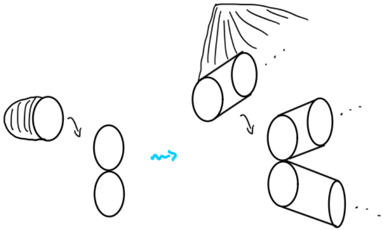
**Example.** The rational space $S^1_{\mathbb{Q}}$ corresponding to a circle.




Higher homotopy groups of this space vanish, and the fundamental group is isomorphic to $\mathbb{Q}$:



The same construction applies to higher dimensional spheres, giving *rationalized spheres*. One could then think of rational spaces in the following way: instead of inductively attaching spheres and taking mapping cones (as we build any cell complex), we attach rationalized spheres and take the mapping cones.



Inclusions of spheres into their rationalized versions induce a rationalization map from any space $X$ to its rationalization $X_{\mathbb{Q}}$.

Alternatively, one can think of the rationalization procedure as "tensoring the Postnikov tower by $\mathbb{Q}$":

$$
\begin{array}{ccc}
 & & S^2 \\
 & & \downarrow \\
 & & \vdots \\
 & & \downarrow \\
K(\mathbb{Z}/2,4) & \longrightarrow & \text{level } 4 \\
 & & \downarrow \\
K(\mathbb{Z},3) & \longrightarrow & \text{level } 3 \\
 & & \downarrow \\
 & & K(\mathbb{Z},2)
\end{array}
$$

Here $K(G,n)$ denotes a space (unique up to homotopy equivalence) which has a single nontrivial homotopy group, $\pi_n(K(G,n)) \cong G$. ("Eilenberg–Maclane spaces")

Alternatively, one can think of the rationalization procedure as "tensoring the Postnikov tower by $\mathbb{Q}$":

$$\begin{tikzcd}
 & S^2 \arrow[r, dashed] \arrow[d] & S^2_{\mathbb{Q}} \arrow[d] & \\
 & \vdots \arrow[d] & \vdots \arrow[d] & \\
K(\mathbb{Z}/2,4) \arrow[r] & \text{level } 4 \arrow[r, dashed] \arrow[d] & \text{level } 4 \arrow[d] & * \arrow[l] \\
K(\mathbb{Z},3) \arrow[r] & \text{level } 3 \arrow[r, dashed] \arrow[d] & \text{level } 3 \arrow[d] & K(\mathbb{Q},3) \arrow[l] \\
 & K(\mathbb{Z},2) \arrow[r, dashed] & K(\mathbb{Q},2) &
\end{tikzcd}$$

Here $K(G, n)$ denotes a space (unique up to homotopy equivalence) which has a single nontrivial homotopy group, $\pi_n(K(G, n)) \cong G$.

The rational cohomology algebra of $K(\mathbb{Q}, n)$ is the free graded–commutative algebra on one generator in degree $n$ (i.e. the polynomial algebra if $n$ is even, and exterior algebra if $n$ is odd); for example $K(\mathbb{Q}, 1) = S^1_{\mathbb{Q}}$ has rational cohomology $\mathbb{Q}[x_1]/(x^2)$, and $K(\mathbb{Q}, 2) = K(\mathbb{Z}, 2)_{\mathbb{Q}} = \mathbb{CP}^{\infty}_{\mathbb{Q}}$ has rational cohomology $\mathbb{Q}[x_2]$.

Knowing this, the previous picture informs us that rational spaces can be thought of as algebras equipped with a differential.

$$\begin{tikzcd} S^2_{\mathbb{Q}} \arrow[d] & K(\mathbb{Q},3) \arrow[l] \quad y_3, d(y_3) = x_2^2 \\ K(\mathbb{Q},2) \quad x_2 & \end{tikzcd}$$

All the topological information of $S^2_{\mathbb{Q}}$ is captured in the free graded–commutative algebra on two generators $x_2$ and $y_3$, equipped with a differential $d$ such that $dx_2 = 0$ and $dy_3 = x_2^2$.

This motivates one to consider Sullivan's approach to rational homotopy theory: to work with rational spaces, one can work with differential graded algebras over $\mathbb{Q}$.

Every dga is equivalent to one which is free and has no linear terms in its differential; from the latter dga one can read off rational topological information of the original space (cohomology, homotopy, Massey products, ...)

If the original space is a smooth manifold, tensoring this construction with $\mathbb{R}$ yields a dga equivalent to the de Rham dga of forms. To *every space* one can associate a dga of piecewise–linear "differential forms", and two spaces have equivalent rationalizations if these dga's are equivalent.

So, which rational spaces $X = X_{\mathbb{Q}}$ are realized by closed almost complex manifolds?

That is, for which $X_{\mathbb{Q}}$ is there a closed almost complex manifold $M$ such that $X_{\mathbb{Q}}$ is its rationalization? What follows will an adaptation to almost complex manifolds of Sullivan's discussion of the smooth case in his 1977 paper "Infinitesimal Computations in Topology".

What are some *necessary* conditions on $X = X_{\mathbb{Q}}$ for there to be such a manifold?

- The rational homology is finite–dimensional.
- The rational cohomology satisfies Poincaré duality, i.e. there is an index $n$ (which would be the dimension of the manifold; in particular $n$ must be even) such that for a choice of non-zero $[X] \in H_n(X;\mathbb{Q}) \cong \mathbb{Q}$, the pairing

  $$H^k(X;\mathbb{Q}) \otimes H^{n-k}(X;\mathbb{Q}) \to \mathbb{Q}$$

  given by

  $$\alpha \otimes \beta \mapsto \langle \alpha\beta, [X] \rangle$$

  is non-degenerate.

More necessary conditions, assuming rational Poincaré duality, will be easier to state if we fix more data: choose a "fundamental class", i.e. a non-zero $[X]$ in top homology, and choose rational "Chern classes" $c_i \in H^{2i}(X; \mathbb{Q})$. We ask:

What are the conditions for there to be an almost complex manifold $(M, [M], J)$ and a map $M \xrightarrow{f} X$ inducing an isomorphism on rational homology, such that $f_*[M] = [X]$ and $f^*(c_i) = c_i(M, J)$ ?

What are the conditions for there to be an almost complex manifold $(M, [M], J)$ and a map $M \xrightarrow{f} X$ inducing an isomorphism on rational homology, such that $f_*[M] = [X]$ and $f^*(c_i) = c_i(M, J)$ ?

- If there were such an $M$, notice that the "Chern numbers" $\langle c_I, [X]\rangle$ satisfy

$$\langle c_I, [X]\rangle = \langle c_I(X), f_*[M]\rangle = \langle f^* c_I(X), [M]\rangle = \langle c_I(M), [M]\rangle = \int_M c_I(M),$$

  and the latter must be integers that satisfy all the congruence conditions coming from the index theorem, i.e.

$$\int \mathrm{ch}(E)\,\mathrm{td}(TM) \in \mathbb{Z} \text{ for any complex vector bundle } E \to M.$$

  Here, recall, $\mathrm{td}$ is the Todd polynomial $\mathrm{td} = 1 + \frac{c_1}{2} + \frac{c_1^2 + c_2}{12} + \frac{c_1 c_2}{24} + \cdots$, and $\mathrm{ch}$ is the Chern character, another rational polynomial in Chern classes. This condition can be recast purely in terms of $X, [X], c_i$.

More precisely, Stong described the image of $\Omega^U \to H_*(BU;\mathbb{Q})$ sending a (stably) almost complex manifold to the pushforward of its fundamental class via the map classifying its (stable) tangent bundle:

To make sense of this statement, let us first review some concepts.

## Cobordism

Two compact manifolds $M, N$ of the same dimension are **cobordant** if there is a compact manifold $W$ *with boundary* whose boundary is the union of $M$ and $N$.

This is an equivalence relation on manifolds; the equivalence classes across all dimensions form a ring, where $+$ is disjoint union or connected sum, and $\times$ is Cartesian product.

**Theorem (Thom)**

*Two compact manifolds are cobordant if and only if they have the same* Stiefel–Whitney numbers*.*

The Stiefel–Whitney numbers are obtained by multiplying Stiefel–Whitney classes up to the top degree and integrating over the manifold.

One can also keep track of orientations in the case of oriented manifolds, and obtain the **oriented cobordism ring** $\Omega^{SO}$.

**Theorem (Wall)**

*Two oriented compact manifolds are oriented cobordant if and only if they have the same Stiefel–Whitney numbers and Pontryagin numbers.*

Now, one would like a notion of **(almost) complex cobordism**, but this will have to involve odd–dimensional manifolds. So let us further relax our notion of almost complex structure.

A **stably almost complex structure** on a manifold $M$ is an almost complex structure on $TM \oplus \mathbb{R}^k$, where $\mathbb{R}^k$ is a trivial real bundle. We refer to $TM \oplus \mathbb{R}^k$ (for any $k \geq 1$) as **the stable tangent bundle** of $M$.

**Example**. All spheres (including the odd–dimensional ones), admit stably almost complex structures.

Now we can talk about the **complex cobordism ring** $\Omega^U$, where there is the following:

**Theorem (Milnor)**

*Two stably almost complex compact manifolds are complex cobordant if and only if they have the same Chern numbers.*

## Grassmannians

Every manifold $M$ can be embedded in a large–dimensional Euclidean space $\mathbb{R}^N$.If this $N$ is very large, any two such embeddings are furthermore homotopic through embeddings. The normal bundle to the manifold in this latter case is independent of the embedding, and we refer to it as the **stable normal bundle**.

Now, look at either the (stable) tangent bundle or normal bundle; denote its rank by $k$. Notice that it gives a map to the Grassmannian of $k$–planes in $\mathbb{R}^n$, by sending a point on $M$ to the corresponding plane at that point. Our bundle is then the pullback via this map of the **tautological bundle** over the Grassmannian, i.e. the bundle over the Grassmannian whose fiber over a point is the $k$–plane that point represents.

Similar reasoning holds for unoriented, oriented, complex, . . . bundles.

Now, suppose $M$ and $N$ are two cobordant manifolds (in any of the above senses) via $W$, with fundamental classes $[M], [N]$. We can consider the map to the Grassmannian which corresponds to the (stable) tangent or normal bundle. The map for $W$ restricts to those for $M$ and $N$ on its boundary. Then since

$$\partial[W] = [M] - [N],$$

the images of $[M]$ and $[N]$ under the corresponding maps are **homologous** in the Grassmannian.

Stong described the image of $\Omega^U \to H_*(BU;\mathbb{Q})$ sending a stably almost complex manifold to the pushforward of its fundamental class via the map classifying its stable tangent bundle:

The image of this map is a lattice in the integral part of the rational homology of the complex Grassmannian $BU$.

The image consists of classes $\alpha$ such that $\langle z \cdot \mathrm{td}, \alpha \rangle \in \mathbb{Z}$ for any $z$ in the integer polynomial ring generated by the elementary symmetric polynomials in the variables $e^{x_i} - 1$, where $x_i$ are the universal Chern roots, $c = (1 + x_1)(1 + x_2) \cdots$.

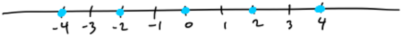
Stably almost complex manifolds correspond to points in this lattice. Stong showed that every point in the lattice is hit.

**Our necessary condition is that the "Chern numbers" of our rational space correspond to a point in this lattice**, i.e. they are the Chern numbers of some genuine stably almost complex manifold.

This reduces to checking *finitely many* universal congruence conditions on the "Chern numbers".

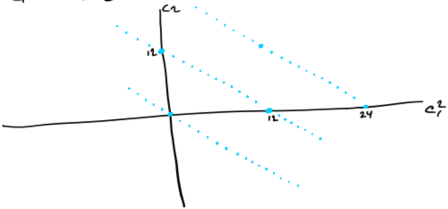
dimension 2

$c_1 \in 2\mathbb{Z}$




dimension 4

$c_1^2 + c_2 \in 12\mathbb{Z}$

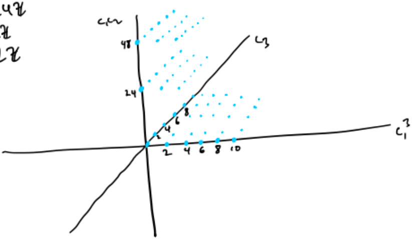



dimension 6

$c_1c_2 \in 24\mathbb{Z}$

$c_3 \in 2\mathbb{Z}$

$c_1^3 \in 2\mathbb{Z}$



What are the conditions for there to be an almost complex manifold $(M, [M], J)$ and a map $M \xrightarrow{f} X$ inducing an isomorphism on rational homology, such that $f_*[M] = [X]$ and $f^*(c_i) = c_i(M, J)$ ?

- Since $\int_M c_n$ is the Euler characteristic of the putative almost complex manifold $M$, we must have $\langle c_n, [X] \rangle$ be the Euler characteristic of $X$.

If the dimension $n$ is divisible by 4, we furthermore have:

- The non-degenerate symmetric bilinear pairing on $H^{\frac{n}{2}}(X; \mathbb{Q})$ must be the rationalization of a unimodular pairing over the integers, i.e. (Milnor–Husemoller) in some basis the pairing on $H^{\frac{n}{2}}(X; \mathbb{Q})$ is of the form $\begin{pmatrix} \pm 1 & & \\ & \ddots & \\ & & \pm 1 \end{pmatrix}$.

- If we form the "Pontryagin classes" via

$$1 - p_1 + p_2 - \cdots = (1 - c_1 + c_2 - \cdots)(1 + c_1 + c_2 + \cdots),$$

  the Hirzebruch $L$–polynomial $1 + \frac{p_1}{3} + \frac{7p_2 - p_1^2}{45} + \cdots$ evaluated on these classes and paired with $[X]$ must calculate the signature of the pairing on $H^{\frac{n}{2}}(X; \mathbb{Q})$ correctly, $\langle L(p_i), [X] \rangle = \sigma(X)$.

As for sufficiency,

**Theorem (Realization for almost complex manifolds, M.)**

*Let $X$ be a simply connected rational space, with $H_*(X;\mathbb{Q})$ finite–dimensional, satisfying rational Poincaré duality, of even dimension $n \geq 6$, with a choice of non-zero $[X] \in H_n(X;\mathbb{Q})$ and a choice of $c_i \in H^{2i}(X;\mathbb{Q})$.*

*Then, if $n \not\equiv 4 \mod 8$, the above necessary conditions*

- *Chern number integrality and congruences,*
- *top Chern class calculating Euler characteristic,*
- *intersection pairing being diagonal with $\pm 1$,*
- *and signature being calculated correctly from the Chern classes,*

*are sufficient for the existence of a closed simply connected almost complex manifold $(M,[M],J)$ and a map $M \xrightarrow{f} X$ inducing an isomorphism on rational homology, such that $f_*[M] = [X]$ and $f^*(c_i) = c_i(M,J)$.*

*If $n \equiv 4 \mod 8$ and $c_1 \neq 0$, the same holds; if $c_1 = 0$, imposing a further congruence condition on the Chern numbers yields the desired result.*

The further congruence in the case of $n \equiv 4 \mod 8$ and $c_1 = 0$ stems from the fact that for an almost complex manifold $M$ with $c_1 = 0$ *integrally* in these dimensions, we have

$$\int_M ch(E \otimes \mathbb{C})\hat{A}(TM) \in 2\mathbb{Z} \text{ for any real vector bundle } E \to M.$$

More precisely, we have a result on realization by *stably* almost complex manifolds, which gives us an almost complex manifold if the top Chern class calculates the Euler characteristic:

**Theorem (M.)**

*Let $X$ be a simply–connected rational space of finite type satisfying Poincaré duality on its rational cohomology, of dimension $n \geq 5$, and let $[X] \in H_n(X;\mathbb{Q})$ be a non-zero element. Let $c_i \in H^{2i}(X;\mathbb{Q})$, $1 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor$ be cohomology classes. Then we have:*

- *If $n$ is odd, there is a closed stably almost complex $n$–manifold $(M,[M],J)$ and a rational equivalence $M \xrightarrow{f} X$ such that $f_*[M] = [X]$ and $c_i(TM) = f^*(c_i)$.*
- *If $n \equiv 2 \bmod 4$, then there is a closed stably almost complex manifold $M$ and a rational equivalence $M \xrightarrow{f} X$ such that $f_*[M] = [X]$ and $c_i(TM) = f^*(c_i)$ if the numbers $\langle c_{i_1} c_{i_2} \cdots c_{i_r}, [X] \rangle$ are integers that satisfy the Stong congruences of a stably almost complex manifold: that is, denoting by $\sigma_i$ the elementary symmetric polynomials in the variables $e^{x_j} - 1$, where the $x_j$ are given by formally writing $1 + c_1 + c_2 + \cdots = \prod_j (1 + x_j)$, we have*

  $$\langle z \cdot td(X), [X] \rangle \in \mathbb{Z} \text{ for every } z \in \mathbb{Z}[\sigma_1, \sigma_2, \ldots].$$

  *Here $td(X)$ denotes the Todd polynomial evaluated on $c_1, c_2, \ldots$.*

## Theorem (cont'd)

- *If $n \equiv 0 \bmod 4$, then there is a closed stably almost complex manifold $M$ and a rational equivalence $M \xrightarrow{f} X$ such that $f_*[M] = [X]$ and $c_i(TM) = f^*(c_i)$ if*
  - *the quadratic form on $H^{\frac{n}{2}}(X;\mathbb{Q})$ given by $q(\alpha,\beta) = \langle \alpha\beta, [X]\rangle$ is equivalent over $\mathbb{Q}$ to one of the form $\sum_i \pm y_i^2$,*
  - *if we define $p_i = (-1)^i \sum_j (-1)^j c_j c_{i-j}$, then $\langle L(p_1,\ldots,p_{n/4}),[X]\rangle = \sigma(X)$, where $L$ is Hirzebruch's $L$–polynomial,*
  - *the numbers $\langle c_{i_1} c_{i_2} \cdots c_{i_r}, [X]\rangle$ are integers that satisfy the Stong congruences of a stably almost complex manifold described above,*
  - *if $c_1 = 0$ and $n \equiv 4 \bmod 8$, the numbers $\langle p_{i_1} p_{i_2} \cdots p_{i_r}, [X]\rangle$ are integers that satisfy a further set of Stong congruences: denoting by $\sigma_i^p$ the elementary symmetric polynomials in the variables $e^{x_j} + e^{-x_j} - 2$, where the $x_j$ are given by formally writing $1 + p_1 + p_2 + \cdots = \prod_j (1 + x_j^2)$, we have*

    $$\langle z \cdot \hat{A}(X), [X]\rangle \in 2\mathbb{Z} \textit{ for every } z \in \mathbb{Z}[\sigma_1^p, \sigma_2^p, \ldots].$$

    *Here $\hat{A}(X)$ denotes the $\hat{A}$ polynomial $\hat{A} = 1 - \frac{p_1}{24} + \frac{7p_1^2 - 4p_2}{5760} + \cdots$ evaluated on $p_1, p_2, \ldots$. Note that these are conditions on $c_1, c_2, \ldots$, as they determine $p_1, p_2, \ldots$.*

*If $\langle c_n(X), [X]\rangle = \chi(X)$, then the stable almost complex structure $J$ on $M$ is induced by a genuine almost complex structure.*

## Elements of the proof.

Take some $N$ that is very large compared to the dimension $n$. The map

$$BU(N) \xrightarrow{(c_1, c_2, \ldots, c_N)} K(\mathbb{Q}, 2) \times K(\mathbb{Q}, 4) \times \cdots \times K(\mathbb{Q}, 2N)$$

picking out the universal rational Chern classes is a rationalization. So is the map

$$BU(N) \xrightarrow{\overline{c}_1, \overline{c}_2, \cdots, \overline{c}_N} K(\mathbb{Q}, 2) \times K(\mathbb{Q}, 4) \times \cdots \times K(\mathbb{Q}, 2N)$$

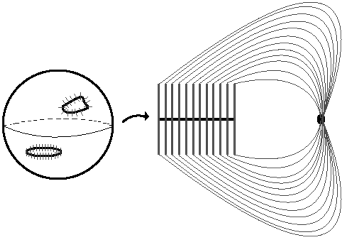
picking out the first $N$ dual Chern classes, i.e. those satisfying $(1 + c_1 + \cdots) \cdot (1 + \overline{c}_1 + \cdots) = 1$.

Consider the homotopy fiber product of this map and the map $X \to K(\mathbb{Q}, 2) \times K(\mathbb{Q}, 4) \times \cdots$ which picks out our chosen Chern classes $c_i(X)$:

$$\begin{tikzcd}
A \arrow[r] \arrow[d] & BU(N) \arrow[d, "\overline{c}_i"] \\
X \arrow[r, "c_i(X)"] & BU(N)_{\mathbb{Q}}
\end{tikzcd}$$

$$
\begin{tikzcd}
A \arrow[r] \arrow[d] & BU(N) \arrow[d, "\overline{c}_i"] \\
X \arrow[r, "c_i(X)"] & BU(N)_{\mathbb{Q}}
\end{tikzcd}
$$

- The map $A \to X$ is an isomorphism on rational homology.
- The advantage of $A$ over $X$ is that we can pull the tautological bundle over $BU(N)$ back over $A$.
- By the Pontryagin–Thom construction we can find closed manifolds mapping to $A$ such that their stable normal bundles are induced by this bundle over $A$, and hence admit complex structures.



- The Chern classes of this complex structure are the duals of those pulled back from $X$.
- The Chern class congruences ensure that we can find a manifold so that the fundamental class pushes forward to $[X]$.
- So we have a degree one map $M \xrightarrow{f} A$ where the stable normal bundle to $M$ is the pullback of the bundle over $A$.
- A non-zero degree map between spaces satisfying rational Poincaré duality is surjective on rational homology. We inductively get rid of the kernel on rational homology $\ker f_*$ by normal surgery (Browder–Novikov):

  by the Hurewicz theorem and Whitney's embedding theorem we represent a basis of the kernel by embedded spheres, which below degree $\frac{n}{2}$ have trivial normal bundle in $M$ by virtue of being in the kernel. We can then remove this sphere and obtain a new manifold $M'$ with degree one map $M' \to A$ inducing the stable normal bundle to $M'$.

  However, in general $A$ will have fundamental group $\mathbb{Q}/\mathbb{Z}$. We either assume $c_1(X) \neq 0$, or if $c_1(X) = 0$ we replace $BU(N)$ by $BSU(N)$ throughout (in which we have to check more congruences on the Chern numbers if $n \equiv 4 \bmod 8$); then $A$ will be simply connected.
- In middle degree (in the case of $n$ even), there are obstructions to performing normal surgery on an embedded sphere which do not vanish a priori: if the middle degree is odd, there is a mod 2 obstruction we can bypass by taking twice the homology class represented by our embedded sphere. If the middle degree is even, the obstruction is $\mathbb{Z}$-valued and vanishes if our conditions on the middle-degree pairing are satisfied (signature and diagonalizable to $\pm 1$).
- The Euler characteristic condition ensures that the complex structure we obtain on the stable tangent bundle is induced by one on the tangent bundle.

So for a given rational space satisfying rational Poincaré duality, one can ask whether it can be equipped with a choice of fundamental class and rational Chern classes, so that the conditions above are satisfied.

- Notice that if the above result lets us conclude that there is an almost complex manifold realizing a given rational space $X$, then since all the conditions are cohomological, *any other rational space $Y$ with isomorphic rational cohomology algebra $H^*(Y;\mathbb{Q}) \cong H^*(X;\mathbb{Q})$ can also be realized*.

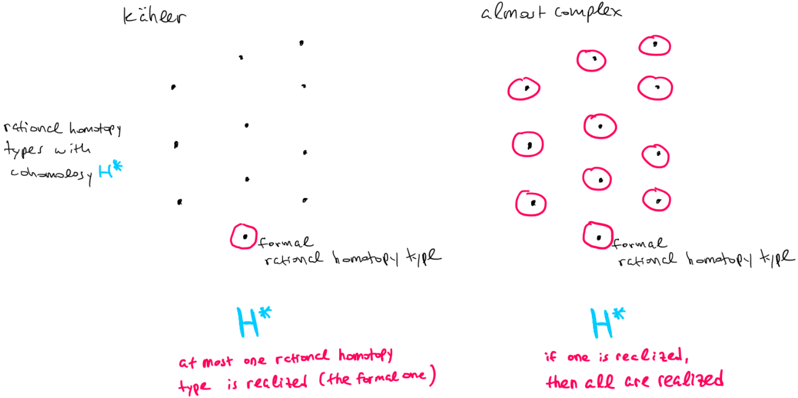
Since rational spaces correspond to commutative differential graded algebras, this allows one to write down many rational homotopy types of almost complex manifolds.

For example, consider the 7–manifold obtained in the following way: take a degree one map $S^2 \times S^2 \to S^4$, and pull back the quaternionic Hopf fiber bundle $S^3 \to S^7 \to S^4$ via this map. Then cross this $S^3$ bundle over $S^2 \times S^2$ to obtain a simply connected 10–manifold.
This has the same rational cohomology as $S^3 \times \left(S^2 \times S^5 \# S^2 \times S^5\right)$, but a different rational homotopy type. Nevertheless, if one of these rational homotopy types is realizable by an almost complex manifold, so is the other.

Contrast this with compact Kähler manifolds, for which we have the following:

**Theorem (Deligne–Griffiths–Morgan–Sullivan)**

*If $X$ and $Y$ are two compact Kähler manifolds with isomorphic rational cohomology ring, then they are rationally homotopy equivalent.*




Are complex manifolds generally somewhere between these two extremes?

**Example.** An almost complex rational cohomology $\mathbb{HP}^3$.

Massey showed that $\mathbb{HP}^n$ with its standard smooth structure does not admit an almost complex structure for any $n$.

Take the rational space $\mathbb{HP}^3_{\mathbb{Q}}$. Its cohomology is $\mathbb{Q}[x]/(x^4)$, where $\deg(x) = 4$.

The dimension is divisible by 4, but the degree 6 part of the algebra is trivial, so the only condition on the middle-degree intersection pairing is that the $L$–polynomial on whatever Chern classes we choose calculates the signature to be 0.

Choose the fundamental class to be such that $x^3$ pairs to 1. Now, $c_1 = c_3 = c_5 = 0$. The Chern numbers must be integers satisfying the following congruences (stated modulo $c_1, c_3, c_5$ terms):

$$
\begin{aligned}
10c_2^3 - 9c_2c_4 + 2c_6 &\in 60480\mathbb{Z} \\
c_2c_4 + 2c_6 &\in 240\mathbb{Z} \\
-c_2^3 + 4c_2c_4 &\in 12\mathbb{Z} \\
c_2^3 - 16c_2c_4 &\in 12\mathbb{Z} \\
c_6 &\in 4\mathbb{Z}
\end{aligned}
\qquad
\begin{aligned}
\tfrac{1}{6048}c_2^3 - \tfrac{1}{6720}c_2c_4 + \tfrac{1}{30240}c_6 &\in 2\mathbb{Z} \\
-\tfrac{1}{120}c_2c_4 - \tfrac{1}{60}c_6 &\in 2\mathbb{Z} \\
-\tfrac{1}{3}c_2^3 + \tfrac{4}{3}c_2c_4 &\in 2\mathbb{Z} \\
\tfrac{1}{12}c_2^3 + \tfrac{1}{3}c_2c_4 + \tfrac{1}{2}c_6 &\in 2\mathbb{Z}
\end{aligned}
$$

and the signature equation translates to

$$5c_2^3 - 36c_2c_4 - 68c_6 = 0.$$

Now, $c_2 = ax, c_4 = bx^2$ for some rational numbers $a, b$, and $c_6 = 4x^3$.

The above congruences simplify to the following Diophantine system:

$$-a^3 + 4ab \in 24\mathbb{Z}$$
$$ab + 8 \in 1920\mathbb{Z}$$
$$5a^3 - 36ab = 248$$

This system has an integer solution of $a = -2$, $b = 4$, and hence we have a simply connected closed almost complex manifold with the same rational homotopy type as $\mathbb{HP}^3$.

This is an almost complex manifold with rational cohomology $\mathbb{Q}[x]/(x^4)$, where $\deg(x) = 4$. Another such manifold, where $\deg(x) = 2$, is $\mathbb{CP}^4$.

One can ask if there are almost complex manifolds with rational cohomology $\mathbb{Q}[x]/(x^k)$, in terms of $k$ and $\deg(x)$:

- For $k = 2$, i.e. **rational homology spheres**, $\deg(x)$ must be 2 or 6, corresponding to the rational homotopy types of $S^2$ and $S^6$. (Albanese–M.) (There are also rational homology 6–spheres that do not admit almost complex structures.)
- For $k = 3$, $\deg(x)$ must be 2, corresponding $\mathbb{CP}^2$. (Albanese–M. for dimensions not a power of 2 and $\geq 2048$, then Jiahao Hu, Zhixu Su for all dimensions).
- For $k = 4$, we see that $\deg(x) = 2, 4$ is realized; unknown for larger $\deg(x)$.

**Example.** Rational connected sums of quaternionic projective planes.

Using a result of Geiges–Müller, one can calculate that $k\mathbb{HP}^2\#\ell\overline{\mathbb{HP}^2}$ (with its standard smooth structure) admits an almost complex structure if and only if $(k,l) = (4n+3, 2n+1)$ for some $n$. Let us see what happens in the rational case; i.e. we consider 8–manifolds $M$ with $H^*(M;\mathbb{Q}) \cong H^*(k\mathbb{HP}^2\#\ell\overline{\mathbb{HP}^2};\mathbb{Q})$. We refer to such a manifold as a *rational* $k\mathbb{HP}^2\#\ell\overline{\mathbb{HP}^2}$.

We will use the Chern number congruences for stably almost complex 8–manifolds:

$$\begin{aligned} -c_4 + c_1c_3 + 3c_2^2 + 4c_1^2c_2 - c_1^4 &\in 720\mathbb{Z}, \\ c_1^2c_2 + 2c_1^4 &\in 12\mathbb{Z}, \\ -2c_4 + c_1c_3 &\in 4\mathbb{Z}, \end{aligned}$$

which in our case trivially becomes

$$-c_4 + 3c_2^2 \in 720\mathbb{Z}, \text{ and } c_4 \text{ is even.}$$

Now, suppose we have a rational $k\mathbb{HP}^2\#\ell\overline{\mathbb{HP}^2}$ that admits an almost complex structure. Then $\sigma = k - \ell$ and $\chi = 2 + k + \ell$, so from Hirzebruch's relation $\sigma \equiv \chi \bmod 4$ in dimension 8, we have $k - \ell \equiv 2 + k + \ell \bmod 4$, i.e. $2\ell \equiv 2 \bmod 4$, i.e. $\ell$ is odd. Since $k + \ell + 2 = \chi = c_4$ must be even, we conclude that $k$ is odd as well.

Consider concretely $k = \ell = 23$.

Then $c_4$ must evaluate to $\chi = 48$, and $\sigma = 0$.

We can write $c_2$ as

$$c_2 = \sum_{i=1}^{23} x_i + \sum_{i=1}^{23} y_i,$$

where the $x_i$ and $y_i$ are degree 4 classes such that $\langle x_i^2, \mu \rangle = 1$ and $\langle y_i^2, \mu \rangle = -1$ for an appropriate choice of fundamental class $\mu$, and $x_i x_j = x_i y_j = y_i y_j = 0$ for all $i \neq j$. (The variables $x_i$ correspond to the degree 4 generators in the $\mathbb{HP}^2$ summands, while the $y_i$ correspond to $\overline{\mathbb{HP}^2}$.)

The signature equation in terms of Chern classes is $\frac{1}{45}(3c_2^2 + 14c_4) = 0$, i.e. $c_2^2 = -224$.

We see that the Stong congruences are satisfied for this $c_2^2$ and $c_4$. Indeed, $c_4$ is even and $-c_4 + 3c_2^2 = 0$. It only remains to check that one can solve for $c_2$. Abstractly, we can use Lagrange's four square theorem for this. Concretely, taking

$$c_2 = 4y_1 + 8y_2 + 12y_3,$$

we have $c_2^2 = -224$.

More generally, we can show there is an almost complex rational $k\mathbb{HP}^2 \# \ell\overline{\mathbb{HP}^2}$ if and only if $(k, l) = (4u + 3, 2u + 1 + 12m)$ with $k, l \geq 0$. The case above with $k = \ell = 23$ is obtained by taking $u = 5$, $m = 1$.

A change of focus: spaces of almost complex structures

Now we know something about what kind of almost complex manifolds are and are not out there. A next question one could ask is:

What does the space of *all* almost complex structures on a given manifold look like?

Let us consider the case of the six–sphere $S^6$, due to its historical interest. The results that follow come from work with Ferlengez and Granja.
We start with some basic considerations:

How many orthogonal (almost) complex structures $J$ are there on $\mathbb{R}^2$ (with respect to the standard inner product)?

There are two: $J$ is determined by where $(1,0)$ is sent: either to $(0,1)$ or to $(0,-1)$.

Though both matrices are in $SO(2)$, the first $J$ induces the counterclockwise orientation on $\mathbb{R}^2$ while the second induces the opposite, clockwise orientation.

So there is only one orthogonal $J$ on $\mathbb{R}^2$ *inducing a given orientation*.

How many orthogonal $J$'s are there on $\mathbb{R}^4$ *inducing the usual orientation*? $S^2$

First, we choose where $1 = (1, 0, 0, 0)$ is sent: this gives us an $S^2$ of options (the unit sphere in the orthogonal complement to $(1, 0, 0, 0)$).

Then, on the $\mathbb{R}^2$ orthogonal to the span of 1 and $J(1)$, there is an induced orientation and hence a unique choice of $J$.

How many orthogonal $J$'s are there on $\mathbb{R}^6$ *inducing the usual orientation*? $\mathbb{CP}^3$

First, choose where $1 = (1,0,0,0,0,0)$ goes: there is an $S^4$ worth of options. Then, on the $\mathbb{R}^4$ orthogonal to the span of 1 and $J(1)$, we have an $S^2$ of almost complex structures.

So the space of orthogonal almost complex structures inducing a given orientation on $\mathbb{R}^6$ is an $S^2$-bundle over $S^4$.

How else can we describe this space? Take the standard $J$ defined by $e_1 \mapsto e_2, e_3 \mapsto e_4, e_5 \mapsto e_6$. We can conjugate this by any element $A \in SO(6)$, and the resulting operator will be an orthogonal almost complex structure also inducing the usual orientation.

This action is in fact transitive on the space of all such almost complex structures, and for $A$ to be in the stabilizer means $AJA^{-1} = J$, i.e. $AJ = JA$, i.e. $A$ is complex linear, $A \in U(3)$.

Hence the space of orthogonal almost complex structures on $\mathbb{R}^6$ inducing the usual orientation is $SO(6)/U(3) \cong \mathbb{CP}^3$.

How many orthogonal $J$'s are there on $\mathbb{R}^8$ *inducing the usual orientation*? $SO(8)/U(4)$

By the previous reasoning, this space is a $\mathbb{CP}^3$-bundle over $S^6$, whose total space is $SO(8)/U(4)$.

Let us look more closely at this picture: what would a section be?

A section determines an *almost complex structure on* $S^6$ inducing the orientation $S^6$ inherits from the usual orientation on $\mathbb{R}^8$.

The space of orthogonal almost complex structures on $S^6$ inducing a given orientation is the space of sections of this bundle $\mathbb{CP}^3 \to SO(8)/U(4) \to S^6$, i.e. we are choosing in each tangent plane an almost complex structure.

If we dropped the "orthogonal" condition, the corresponding bundle would have fiber $GL^+(6,\mathbb{R})/GL(3,\mathbb{C})$.

The inclusions

$$\left\{\text{smooth sections of the } \mathbb{CP}^3 \text{ bundle}\right\} \hookrightarrow \left\{\text{continuous sections of the } \mathbb{CP}^3 \text{ bundle}\right\}$$

and

$$\left\{\text{continuous sections of the } \mathbb{CP}^3 \text{ bundle}\right\} \hookrightarrow \left\{\text{continuous sections of the } GL^+(6,\mathbb{R})/GL(3,\mathbb{C}) \text{ bundle}\right\}$$

are (weak) homotopy equivalences by smooth approximation arguments, so we can move between the various bundles for computation's sake.

## Almost complex structures provided by the octonions

Thinking of $S^6 = \{x^2 + 1 = 0\}$ as the unit sphere in the imaginary octonions (i.e. unit length octonions with real part $= 0$), we can produce an orthogonal metric-compatible $J$ as follows:

$J_p(v) = pv$

Why does this work? Any real subalgebra of the octonions generated by two elements *is associative* (Artin, Frobenius).

So we have $J_p(J_p(v)) = p(pv) = (pp)v = -v$. Is $pv \in T_pS^6 = \mathrm{span}(1, p)^\perp$? Indeed, $\langle pv, 1\rangle = \langle ppv, p\rangle = \langle -v, p\rangle = 0$, and similarly $\langle pv, p\rangle = \langle v, 1\rangle = 0$.

Can we get more from $J_p(v) = pv$?

We can transport any orthogonal metric-compatible almost complex structure $J$ around by any isometry $A \in SO(7)$ of $S^6$ to get another (possibly the same) such almost complex structure, as follows:

$$(A.J)_p(v) = A^{-1}(J_{Ap}(Av))$$

So $SO(7)$ acts on the space of such almost complex structures. What does the orbit of the octonionic $J_p(v) = pv$ look like? What is the stabilizer?

If $A.J = J$ for $A \in SO(7)$, then $(A.J)_p(v) = A^{-1}(J_{Ap}(Av)) = A^{-1}((Ap)(Av))$ has to equal $J_p(v) = pv$, so $A^{-1}((Ap)(Av)) = pv$, i.e.

$$(Ap)(Av) = A(pv).$$

This is enough to conclude that $A$ is a real-algebra automorphism of the octonions, i.e. it is in the group $G_2$. So the orbit of $J_p(v) = pv$ under the $SO(7)$ action is $SO(7)/G_2 \cong \mathbb{RP}^7$.

This orbit $SO(7)/G_2 \cong \mathbb{RP}^7$ can be described explicitly (Battaglia) as those $J$ given by $J_p(v) = (p(vx))\overline{x}$, where $x \in S^7$ is a unit octonion.

**Question**: Is the inclusion $\mathbb{RP}^7 \hookrightarrow \{\text{all almost complex structures}\}$ a homotopy equivalence?

**Motivation**: Rationally, one can calculate using the Haefliger–Sullivan rational model for the space of sections of a fiber bundle, that the space of sections of our $\mathbb{CP}^3$ bundle has the rational homology/homotopy of $S^7$, or equivalently $\mathbb{RP}^7$.

## Another $\mathbb{RP}^7$?

Before we consider the inclusion
$SO(7)/G_2 = \mathbb{RP}^7 \hookrightarrow \{\text{all almost complex structures }\}$, notice the following:

For any unit octonion $x \in S^7$, we can consider conjugation by $x$ as an element of $SO(8)$, i.e. $a \mapsto xa\bar{x}$. Since this fixes the real line, we get a map $S^7 \to SO(7)$, and it factors through $\mathbb{RP}^7$.

So we have a map $\mathbb{RP}^7 = S^7/(\mathbb{Z}/2) \to SO(7)/G_2 = \mathbb{RP}^7$. What does this map look like?

To begin with, what is the preimage of $[G_2] \in SO(7)/G_2$, i.e. *which unit octonions $x$ are such that conjugation by them is multiplicative*?

These are precisely the sixth roots of unity $\{x^6 = 1\}$. (Brandt, Zorn)

Generically, this map $\mathbb{RP}^7 \to \mathbb{RP}^7$ is 3-to-1, induced by $x \mapsto ((p, v) \mapsto (p(v\overline{x}^3))x^3)$.
Hence it induces an isomorphism on fundamental groups and on rational homology/homotopy.

## Fundamental group of the space of almost complex structures.

From now on let $J(S^6)$ denote the space of sections of the bundle $\mathbb{CP}^3 \to SO(8)/U(4) \to S^6$, i.e. the space of (metric and orientation compatible) almost complex structures.

We will calculate $\pi_1(J(S^6))$ using the following result of Crabb–Sutherland:

**Theorem (Crabb–Sutherland)**

*Let $X$ be a closed connected oriented $2n$–manifold and $\xi$ a complex $(n+1)$-plane bundle over $X$. Denote by $N\xi$ the space of sections of the projective bundle $\mathbb{P}(\xi)$ that lift to sections of $\xi$ (that are in the same connected component as some given section). Then $\pi_1(N\xi)$ is a central extension*

$$0 \to \mathbb{Z}/c_n(\xi)[X] \to \pi_1(N\xi) \to H^1(X;\mathbb{Z}) \to 0.$$

We can apply this to our situation of $X = S^6$ *if our $\mathbb{CP}^3$ bundle were the projectivization of some rank 4 complex vector bundle $\xi$.*

Luckily, it is the case that our $\mathbb{CP}^3$ bundle is the projectivization of a rank 4 complex vector bundle, namely the bundle of *positive pure spinors*. (see Lawson–Michelsohn "Spin geometry")

From obstruction theory we see that $J(S^6)$, i.e. the space of sections of the projectivized bundle $\mathbb{CP}^3 \to SO(8)/U(4) \to S^6$ is non-empty (as we already knew) and *connected*.

We also see that every section of the projectivization of $\xi$ lifts to a section of $\xi$ (the obstructions to finding a section of a circle bundle vanish over $S^6$ as $H^2(S^6;\mathbb{Z}) = 0$).

Finally, since $H^1(S^6;\mathbb{Z}) = 0$, we conclude from Crabb–Sutherland that $\pi_1(J(S^6)) = \mathbb{Z}/c_3(\xi)[S^6]$. We only have to calculate $c_3$ of the bundle of positive spinors.

We calculate $c_3$ of the bundle of positive spinors by using the projective bundle formula on the projective bundle $\mathbb{CP}^3 \to SO(8)/U(4) \xrightarrow{p} S^6$:

$H^*(SO(8)/U(4); \mathbb{Z})$ is a free $p^* H^*(S^6; \mathbb{Z})$-module generated by a single degree 2 element $x$, subject only to the relation

$$x^4 + c_1(\xi)x^3 + c_2(\xi)x^2 + c_3(\xi)x + c_4(\xi) = 0.$$

From this and knowledge of $H^*(SO(8)/U(4); \mathbb{Z})$ we can conclude $c_3(\xi)[S^6] = \int_{S^6} c_3(\xi) = \pm 2$, and so $\pi_1(J(S^6)) = \mathbb{Z}/2$.

## Higher homotopy groups of the space $J(S^6)$ of almost complex structures.

To get at the higher homotopy groups of $J(S^6)$, we consider a relative Postnikov tower for the map $SO(8)/U(4) \to S^6$:

$$\begin{array}{ccc}
 & & SO(8)/U(4) \\
 & & \big\downarrow p \\
 & & \vdots \\
 & & \big\downarrow \\
K(\mathbb{Z}/2, 8) & \longrightarrow & E_3 \\
 & & \big\downarrow p_3 \\
K(\mathbb{Z}, 7) & \longrightarrow & E_2 \\
 & & \big\downarrow p_2 \\
K(\mathbb{Z}, 2) & \longrightarrow & E_1 \\
 & & \big\downarrow p_1 \\
 & & S^6
\end{array}$$

The spaces being fibered in are the fibers in the Postnikov tower of $\mathbb{CP}^3$.

Haefliger describes a similar decomposition into a sequence of fibrations for the associated space of sections $J(S^6)$. Let $\Gamma_i$ denote the space of sections of $E_i \to S^6$. $Map(S^6, -)$ denotes the space of (unbased) maps from $S^6$ to the target.

$$
\begin{tikzcd}
 & SO(8)/U(4) \arrow[d, "p"] & & J(S^6) \arrow[d] \\
 & \vdots \arrow[d] & & \vdots \arrow[d] \\
K(\mathbb{Z}/2, 8) \arrow[r] & E_3 \arrow[d, "p_3"] & Map(S^6, K(\mathbb{Z}/2, 8)) \arrow[r] & \Gamma_3 \arrow[d] \\
K(\mathbb{Z}, 7) \arrow[r] & E_2 \arrow[d, "p_2"] & Map(S^6, K(\mathbb{Z}, 7)) \arrow[r] & \Gamma_2 \arrow[d] \\
K(\mathbb{Z}, 2) \arrow[r] & E_1 \arrow[d, "p_1"] & & \Gamma_1 \\
 & S^6 & &
\end{tikzcd}
$$

The bundle $K(\mathbb{Z}, 2) \to E_1 \to S^6$ is trivial since it is classified by a map $S^6 \to BK(\mathbb{Z}, 2) = K(\mathbb{Z}, 3)$, and $H^3(S^6; \mathbb{Z}) = 0$. So $E_1 = S^6 \times K(\mathbb{Z}, 2)$ and $\Gamma_1 = Map(S^6, K(\mathbb{Z}, 2))$.

By a result of Thom, the space of maps into an Eilenberg–Maclane space is a product of Eilenberg–Maclane spaces, depending only on the cohomology groups of the domain space. In our case, we have $Map(S^6, K(G,n)) = K(G,n) \times K(G, n-6)$, where if $n - 6 < 0$ the second factor is a point. So we have the sequence of fibrations:

$$\begin{tikzcd}
 & J(S^6) \arrow[d] \\
 & \vdots \arrow[d] \\
K(\mathbb{Z}/2, 8) \times K(\mathbb{Z}/2, 2) \arrow[r] & \Gamma_3 \arrow[d] \\
K(\mathbb{Z}, 7) \times K(\mathbb{Z}, 1) \arrow[r] & \Gamma_2 \arrow[d] \\
 & K(\mathbb{Z}, 2)
\end{tikzcd}$$

Now using $\pi_1(J(S^6)) = \mathbb{Z}/2$, we have enough information to determine $\pi_2(J(S^6))$:

$$
\begin{tikzcd}
 & J(S^6) \arrow[d] \\
 & \vdots \arrow[d] \\
K(\mathbb{Z}/2,8) \times K(\mathbb{Z}/2,2) \arrow[r] & \Gamma_3 \arrow[d] \\
K(\mathbb{Z},7) \times K(\mathbb{Z},1) \arrow[r] & \Gamma_2 \arrow[d] \\
 & K(\mathbb{Z},2)
\end{tikzcd}
$$

$$\begin{tikzcd}
 & J(S^6) \arrow[d] \\
 & \vdots \arrow[d] \\
K(\mathbb{Z}/2, 8) \times K(\mathbb{Z}/2, 2) \arrow[r] & \Gamma_3 \arrow[d] \\
K(\mathbb{Z}, 7) \times K(\mathbb{Z}, 1) \arrow[r] & \Gamma_2 \arrow[d] \\
 & K(\mathbb{Z}, 2)
\end{tikzcd}$$

We get $\pi_2(J(S^6)) = \mathbb{Z}/2$.

So, since $\pi_2(\mathbb{RP}^7) = 0$, the space of almost complex structures does not have the homotopy type of $\mathbb{RP}^7$.

What else can we say about the homotopy type, and about the map

$$\mathbb{RP}^7 \hookrightarrow \{\text{all almost complex structures}\}?$$

Consider the $S^1$ action on $S^6 \times S^7$ given by $e^{i\theta} \cdot (p, x) = (p, (\cos\theta + p\sin\theta)x)$. Then this induces a diffeomorphism $(S^6 \times S^7)/S^1 \cong SO(8)/U(4)$ given by

$$[(p, x)] \mapsto (v \mapsto (p(vx))\bar{x}).$$

We can then consider the map of fiber bundles

$$\begin{tikzcd} S^6 \times S^7 \arrow[r] \arrow[d] & SO(8)/U(4) \arrow[d] \\ S^6 \arrow[r, "="] & S^6 \end{tikzcd}$$

which, upon evaluating at e.g. $i \in S^6$, induces the map of fibrations

$$\begin{tikzcd} Map(S^6, S^7) \arrow[r] \arrow[d, "ev"] & J(S^6) \arrow[d, "ev"] \\ S^7 \arrow[r] & \mathbb{CP}^3 \end{tikzcd}$$

which we can complete to a 3-by-3 diagram of fibrations:

$$
\begin{tikzcd}
Map_*(S^6, S^7) \arrow[r] \arrow[d] & J_*(S^6) \simeq Map_*(S^6, \mathbb{CP}^3) \arrow[d] \\
Map(S^6, S^7) \arrow[r] \arrow[d, "ev"] & J(S^6) \arrow[d, "ev"] \\
S^7 \arrow[r] & \mathbb{CP}^3
\end{tikzcd}
$$

$$\begin{tikzcd}
\Omega^6 S^7 \arrow[r] \arrow[d] & \Omega^6 \mathbb{CP}^3 \arrow[d] \\
(\Omega^6 S^7) \times S^7 \arrow[r] \arrow[d, "ev"] & J(S^6) \arrow[d, "ev"] \\
S^7 \arrow[r] & \mathbb{CP}^3
\end{tikzcd}$$

$$
\begin{tikzcd}
\Omega^6 S^1 \simeq * \arrow[r] & \Omega^6 S^7 \arrow[r] \arrow[d] & \Omega^6 \mathbb{CP}^3 \arrow[d] \\
& (\Omega^6 S^7) \times S^7 \arrow[r] \arrow[d, "ev"] & J(S^6) \arrow[d, "ev"] \\
S^1 \arrow[r] & S^7 \arrow[r] & \mathbb{CP}^3
\end{tikzcd}
$$

$$
\begin{tikzcd}
* \arrow[r] \arrow[d] & \Omega^6 S^7 \arrow[r] \arrow[d] & \Omega^6 \mathbb{CP}^3 \arrow[d] \\
S^1 \arrow[r] \arrow[d] & (\Omega^6 S^7) \times S^7 \arrow[r] \arrow[d, "ev"] & J(S^6) \arrow[d, "ev"] \\
S^1 \arrow[r] & S^7 \arrow[r] & \mathbb{CP}^3
\end{tikzcd}
$$

Now, knowing that $\pi_1 J(S^6) = \mathbb{Z}/2$, this is enough to obtain

$$\pi_k J(S^6) = \pi_k S^7 \oplus \pi_{k+6} S^7 \text{ for } k \geq 2.$$

Similarly, we can consider the map of fibrations

$$\begin{tikzcd} \mathbb{RP}^7 \arrow[r, hook] \arrow[d, "ev"] & J(S^6) \arrow[d, "ev"] \\ \mathbb{CP}^3 \arrow[r, "="] & \mathbb{CP}^3 \end{tikzcd}$$

extend it to a 3-by-3 diagram of fibrations, and see that the homotopy fiber of $\mathbb{RP}^7 \hookrightarrow J(S^6)$ has the homotopy type of a connected component of $\Omega^7 S^7$.

Hence the induced map $\pi_1 \mathbb{RP}^7 \hookrightarrow \pi_1 J(S^6)$ is a surjection, and thus an isomorphism.

Furthermore, since $\pi_{\geq 1} \Omega^7 S^7 \otimes \mathbb{Q} = 0$, the inclusion $\mathbb{RP}^7 \hookrightarrow J(S^6)$ induces an isomorphism on all rational homotopy groups (and on rational homology).

Alternative argument for rational homotopy isomorphism.

Let us take as known that $J(S^6)$ has a single non-trivial rational homotopy group, namely $\pi_7 J(S^6) \otimes \mathbb{Q} = \mathbb{Q}$, and give an alternative argument that the inclusion $\mathbb{RP}^7 = SO(7)/G_2 \hookrightarrow J(S^6)$ is an isomorphism on rational homotopy groups.

Consider the evaluation map $(SO(7)/G_2) \times S^6 \xrightarrow{ev} SO(8)/U(4)$. Calabi and Gluck described this map upon identifying $SO(8)/U(4)$ with the Grassmannian $Gr^+(2,8)$ of oriented real 2–planes in $\mathbb{R}^8$:

a fixed $J \in SO(7)/G_2 = \mathbb{RP}^7$ sends $S^6$ to the sub-Grassmannian of two–planes containing a fixed line in $\mathbb{R}^8$; the space of such lines is parametrized by the $\mathbb{RP}^7$ of considered almost complex structures.

From this description, we can conclude that the evaluation map is a fiber bundle with fiber $S^1$.

$$
\begin{tikzcd}
S^1 \arrow[r] & (SO(7)/G_2) \times S^6 \arrow[rr, "ev"] \arrow[dr, hook, "(\iota, id)"'] & & SO(8)/U(4) \\
& & J(S^6) \times S^6 \arrow[ur, "ev"'] &
\end{tikzcd}
$$

From the homotopy sequence for the fibration $\mathbb{CP}^3 \to SO(8)/U(4) \to S^6$ we see $\pi_7(SO(8)/U(4)) \otimes \mathbb{Q} = \mathbb{Q}$, and so from the above diagram we conclude the map $\mathbb{RP}^7 \to J(S^6)$ induces an isomorphism on $\pi_7(-) \otimes \mathbb{Q}$.

## Intersecting almost complex structures

Notice that a section $s$ of the twistor space $\mathbb{CP}^3 \to SO(8)/U(4) \to S^6$, i.e. an orthogonal almost complex structure on $S^6$, gives us an embedding of $S^6$ in $SO(8)/U(4)$.

We can consider the homological self-intersection number of the corresponding homology class $s_*[S^6]$. To obtain this, one either perturbs $s(S^6)$ to a tranverse submanifold and counts the intersections points (with multiplicity), or squares the Poincaré dual of $s_*[S^6]$ and integrates over $SO(8)/U(4)$.

In general one can consider the "twistor space" over a Riemannian 6–manifold, with fiber $\mathbb{CP}^3$. Sections of this correspond to orthogonal almost complex structures. In this general setup we have:

**Theorem (Granja–M.)**

*Let $(M^6, g)$ be a closed Riemannian 6–manifold, equipped with an almost complex structure $J$ with Chern classes $c_i$. Then the homological self-intersection number of the section $s$ of the twistor space corresponding to $J$ is*

$$\int_M c_1 c_2 - c_3.$$

For example, for $S^6$ we get that this number is $-2$. Notice that this is *metric-independent*; but one must choose a metric to calculate it.

The total space of the twistor bundle has a "canonical" almost complex structure (see Lawson–Michelsohn "Spin Geometry").

This almost complex structure is particularly nice because of the following:

**Theorem (Michelsohn)**

*An almost complex structure $J$ on $M$ is integrable if and only if the section $s$ it corresponds to is a pseudoholomorphic map with respect to $J$ and the canonical almost complex structure on the twistor space.*

Being pseudoholomorphic means $(ds) \circ J = J \circ (ds)$ (i.e., it is what would be called a holomorphic map, but between almost complex manifolds).

Now suppose $S^6$ (or any other 6–manifold) had an integrable complex structure compatible with a fixed metric. **Then $s(S^6)$ would be a pseudoholomorphically embeddded (almost) complex submanifold of the twistor space**.

If we could perturb $s(S^6)$ to **another almost complex submanifold** of the twistor space **intersecting this one transversally**, then the homological self-intersection would have to be $\geq 0$ since this is the intersection of two almost complex submanifolds in an almost complex manifold.

**Example**. Fix the round metric on $S^6$, that is, the one it inherits from $\mathbb{R}^7$ and the usual embedding.

The $\mathbb{RP}^7$ of almost complex structures we considered earlier, coming from octonions, are orthogonal with respect to the round metric.

Calabi–Gluck showed how one can, in the case of the round metric, think of the twistor space as the Grassmannian of oriented 2–planes in $\mathbb{R}^8$, and this $\mathbb{RP}^7$ corresponds to embedding $S^6$ into this Grassmannian as the sub-Grassmannian of 2–planes containing a fixed line.

Two such embeddings intersect transversally. If one of them were integrable, they would all be integrable (since they are obtained from one another by isometries); since the homological self-intersection of any almost complex structure on $S^6$ is $-2$, this shows these almost complex structures are *not* integrable.

There are of course other ways to see this, but perhaps this technique yields other results.